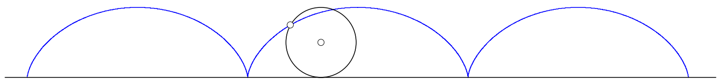
Imagine a circle rolling along a line like a wheel rolls along a road. As the circle rolls, a point on the circle traces a path called a *cycloid*. In this activity you will use parametric functions to define and plot a cycloid.



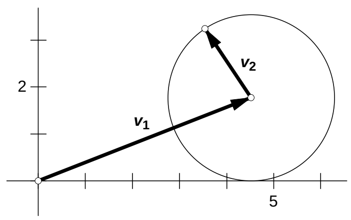
1. Open the Geometric page of **Cycloid.gsp** in the folder **Supplemental Activities | Cycloid**.

The cycloid you create in this activity will be defined parametrically, not geometrically.

This page contains a completed geometric construction of a cycloid, to give you a clear picture of the curve you will define using parametric functions. Your functions will model a circle rolling on the *x*-axis, starting from the origin.

Solving a problem by breaking it into parts is called *decomposition*.

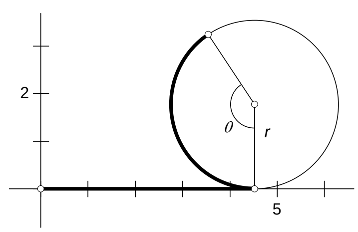
The trick for defining the functions is to break the problem into parts using the principle of vector addition. You will define one vector $\boldsymbol{v}_1$ from the origin to the center of the circle. You will define a second vector $\boldsymbol{v}_2$ from the center point to the point being traced. The sum of these vectors represents the position of the traced point.




2. Open the Parametric page of **Cycloid.gsp**.

There are also several action buttons that you will use later.

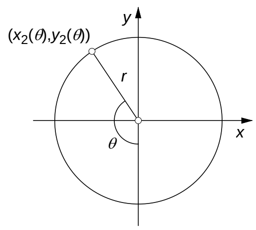
Some of the objects you will need have been prepared for you. The value *r* is the radius of the circle, and *cycles* is the number of turns the wheel will make. You'll use measurement $\theta$ (measured in radians) as the parameter. You will modify functions $x_1(\theta)$ and $y_1(\theta)$ to be the components of vector $\boldsymbol{v}_1$, and $x_2(\theta)$ and $y_2(\theta)$ to be the components of $\boldsymbol{v}_2$. A third pair of functions, $x(\theta)$ and $y(\theta)$, is already defined to be the sum of the other two.




Q1 After the circle has turned to the right by angle $\theta$, the *x*-coordinate of the center point is equal to the length of the line segment over which the circle has rolled. This, in turn, must be equal to the length of the circle arc that rolled over it. What is that length? Edit function $x_1(\theta)$ so it has this value.

Q2 What is the *y*-coordinate of the center of the circle? Edit $y_1(\theta)$ to have this value.

Your answers to Q1 and Q2 define vector $\boldsymbol{v_1}$. To define $\boldsymbol{v_2}$, imagine the center point of the circle as the origin of another coordinate system. The traced point starts at the bottom of the circle and moves around in a clockwise direction as the circle turns.




Q3 In this coordinate system, what is the $x$-coordinate of the traced point? Edit $x_2(\theta)$ so it has this value.

Q4 What is the $y$-coordinate of the traced point with respect to the center of the circle? Edit $y_2(\theta)$ to have this value.

The calculations appear as *x(theta)* and *y(theta)*.

3. Calculate $x(\theta)$ and $y(\theta)$. Plot these coordinates. Press the *Animate* button and observe the motion of the plotted point.

## PRESENT

The driver point is originally hidden. Use the *Show driver* button to show it.

4. Draw the cycloid by tracing the plotted point while the animation continues.
5. To construct the cycloid as a locus, select the driver point and the plotted point. Then choose **Construct | Locus**.

To plot the center point, you will need to calculate $x_1$(*theta*) and $y_1$(*theta*).

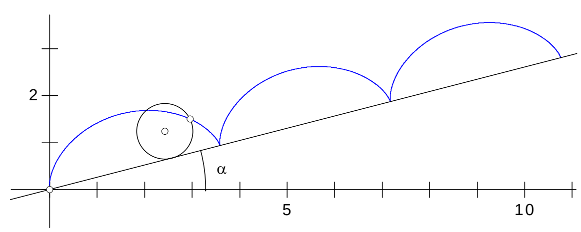
6. The construction will be easier to understand if you can see the circle. Plot the center point (the point determined by vector $\boldsymbol{v_1}$) and then construct the circle.
7. Use the text tool to present the definition of the curve as a single pair of parametric functions.

## EXPLORE MORE

8. Make another cycloid with the circle rolling to the left.
9. Make it roll up the $y$-axis.

Hint: the solution to this challenge is a rotation of the original construction.

10. Define a parameter *alpha*. Make the circle roll along an inclined line intersecting the $x$-axis at angle *alpha*.



# CYCLOID

**Objective:** Students use vector addition to define parametric functions for a cycloid.

**Prerequisites:** Students should have a good understanding of parametric functions, radian angle measurement, and elementary trigonometry. Though vector addition is used, students should be able to grasp its application here even if they haven't studied vectors yet.

**Sketchpad Proficiency:** Intermediate. A partially completed sketch has been provided to save time. Students edit functions and perform some simple constructions, such as plotting points and constructing a locus.

**Class Time:** 30–40 minutes for the basic construction and presentation. The Explore More section includes three variations that you can assign if enough time is available.

**Required Sketch: Cycloid.gsp**

**Example Sketch: Cycloid Work.gsp**

The activity begins with students observing a completed geometric cycloid construction. This geometric construction is provided in order to give them a clear picture of the curve before beginning their own work.

The answers to the four questions are simply the function definitions.

Q1 $x_1(\theta) = r\theta$

Q2 $y_1(\theta) = r$

Q3 $x_2(\theta) = -r\sin(\theta)$

Q4 $y_2(\theta) = -r\cos(\theta)$

## PRESENT

In step 4 students construct a locus by first showing and selecting the hidden driver point. This could be a bit difficult for students to follow if it is not modeled for them. If there is still trouble, trace the point instead.

The presentation should include a clear statement of the compound parametric functions:

$x(\theta) = r(\theta - \sin\theta)$

$y(\theta) = r(1 - \cos\theta)$

## EXPLORE MORE

This section challenges students to construct three variations on the cycloid. All three can be constructed by editing the work that students just completed. The last one is more difficult than the others; don't expect students to do it without hints and guidance.

The final parametric functions are shown below. These will render the cycloid, but in order to construct the rolling circle, the functions will have to be defined in parts, as before.

8. To make a cycloid with the circle rolling to the left, change the sign of $x$:

   $x(\theta) = r(\sin\theta - \theta)$

   $y(\theta) = r(1 - \cos\theta)$

9. To make it roll up the $y$-axis, switch the definitions for $x$ and $y$:

   $x(\theta) = r(1 - \cos\theta)$

   $y(\theta) = r(\theta - \sin\theta)$

10. To make the circle roll along an incline, start with the functions that make it roll along the $x$-axis. Then rotate the combined vector (the sum of $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$) by the angle $\alpha$, using this transformation matrix:

    $$\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

    The rotation yields these functions:

    $x(\theta) = r(\theta - \sin\theta)\cos\alpha - r(1 - \cos\theta)\sin\alpha$

    $y(\theta) = r(\theta - \sin\theta)\sin\alpha + r(1 - \cos\theta)\cos\alpha$

    To plot the center point of the rolling circle, apply the same transformation to vector $\boldsymbol{v_1}$ by itself.