Barnsley's fern is a fractal created by an *iterated function system*, in which a point (the *seed* or *pre-image*) is repeatedly transformed by using one of four transformation functions. A random process determines which transformation function is used at each step. The final image emerges as the iterations continue.

Affine transformations preserve collinearity and ratios of distances.

The transformations are affine transformations of form

$$x' = ax + cy + e$$

$$y' = bx + dy + f$$

Thus each transformation can be specified by six constants (a, b, c, d, e, and f in the example).

SKETCH

1. Open the sketch **Fractal Fern.gsp** in the folder **Supplemental Activities | Fractal Fern**. This sketch contains the constants to be used in the four transformations and tools to simplify the construction process.

A random process controls which transformation is chosen for each iteration. For the Barnsley fern, the first transformation is chosen 1% of the time, each of the next two is chosen 7% of the time, and the fourth transformation is chosen 85% of the time. You'll start out by constructing a point that can be used to make the random choices.

To measure the ratio, select *A*, *B* and *C* in order. Then choose **Measure | Ratio.**

2. Construct point C on horizontal segment AB and measure the three-point ratio AC/AB. Label the ratio r. Drag C to be sure that r stays between 0 and 1.
3. To assign probabilities to each of the transformations, create five parameters t_1 through t_5. These parameters define the range of values associated with each transformation. To achieve probabilities of 1%, 7%, 7% and 85%, assign the parameters values of 0.00, 0.01, 0.08, 0.15, and 1.00. Each successive pair of parameters defines the range of values for one of the transformations.

The Between tool performs a calculation whose result is 1 when a test value is between two reference values, and whose result is 0 otherwise. To choose this custom tool, click and hold the **Custom Tool** icon, and choose **Between** from the menu that appears.

4. Choose the **Between** custom tool and use it to generate results m_1 through m_4, using r as the test value and pairs of parameters t_1 through t_5 as the reference values. (In other words, m_1 should produce 1 when r is between t_1 and t_2, and should produce 0 otherwise. Similarly, m_2 should produce 1 only when r is between t_2 and t_3.) Drag C to make sure that, no matter where C is, exactly one of the values m_1 through m_4 produces 1, and the other three produce 0.
5. Make sure the origin is near the bottom center of the screen, and that the scale of the axes leaves 10 on the y-axis slightly below the top of the screen.

Make sure these labels are correct, or the **Affine Transformation** tool won't work correctly.

6. Construct an independent point P to be the pre-image for the iteration, and measure its x- and y-coordinates. Label the coordinates x and y (no subscripts).

7. Choose the **Affine Transformation** tool, and use it to generate the four transformation functions. For each function, click on the appropriate values for a, b, c, d, e, and f.

This calculation will compute the x value corresponding to the current value of r.

8. To calculate a transformed x value from the four functions, create a calculation that adds the x values of all four transformations, multiplying each x value by the corresponding value from m_1 through m_4.

$$m_1\,(a_1x + c_1y + e_1) + m_2\,(a_2x + c_2y + e_2) + m_3\,(a_3x + c_3y + e_3) + m_4\,(a_4x + c_4y + e_4)$$

Point Q is the transformed image of P.

9. Calculate a y value in the same way. Plot the point determined by these two calculations, and label it Q.

As you drag C, you should observe Q switching among four possible transformed images.

10. The value of r determines which transformation is used. Drag C to change the value of r. Observe that Q moves as each of the transformations is used in turn.

You'll iterate the construction by iterating P to Q and by moving images of C to new random positions on the segment.

11. Iterate the function system: select points P and C. Choose **Transform | Iterate**, and iterate P to Q and C to itself. Click the Iterate button.

12. Delete the table of iterated values that appears. Choose **Edit | Properties | Iteration** for the iterated image of P, and set the number of iterations to 20. Choose **Display | Line Width | Dashed** to show the image with small points.

Q1 Observe that the orbit of the iterated images tends toward a fixed point. Drag point C to change the transformation used. Does each transformation function appear to be associated with a single fixed point? Give the approximate coordinates of the fixed point corresponding to each function.

13. The initial orbit that appears uses only a single transformation, because the images of C are not yet taking on new random positions. Select the iterated image of P and choose **Edit | Properties | Iteration**. Change the number of iterations to 1000, and click the radio button labeled To Random Locations On Iterated Paths.

The 1000 iterations that appear produce an outline of the final shape, but you'll need many more iterations to develop it in detail. One way to generate more detail is to trace the iterated image of P while rerandomizing the positions of images of C.

14. Select point C and choose **Display | Animate**.

To trace, choose **Display | Trace Iterated Point.**

15. Select the iterated image of P and turn on tracing. Then open Iteration Properties, and click the Randomize Locations button. Each time you click the button, the images of C are assigned new random locations, and new images of P leave traces on the screen. Continue pressing the Randomize Locations button

until you have a fair amount of detail in the image of the fern. Once you have enough detail to see the fern clearly, click OK to dismiss the dialog box.

16. Stop the animation of point *C*, and then drag *C* so the fourth function is used to generate point *Q* from *P*.

Q2 Move *P* to different locations on the fern, and describe the effect of this transformation function in terms of the shape of the fern.

Q3 Similarly, activate each of the other transformation functions, and describe the effect of these functions.

To find a fixed point of a transformation, drag *P* until *P* and *Q* coincide.

Q4 Where are the fixed points of the four transformations in terms of the shape of the fern?

Q5 Change the *t* parameters so the first transformation is never used. Erase the traces and regenerate the fern using only transformations 2, 3, and 4. What's the difference in the new shape? Describe the role of the first transformation.

Q6 Determine and describe the role of each of the other transformations in generating the image.

EXPLORE MORE

Q7 Determine the effect of the various constants in specifying the fern's shape.

Q8 Modify the fern transformations so that the leaves of the fern are opposite each other, instead of alternating.

Q9 Make other minor changes to the function system to produce a modified shape. (Be prepared to undo your changes – not all function systems generate interesting results.)

Similar iterated function systems can generate a wide variety of fractals. For instance, consider a system of three functions that have the following effects:

- Dilate the pre-image point by a scale factor of one-half toward the point (6, 0).
- Dilate the pre-image point by a scale factor of one-half toward the point (-6, 0).
- Dilate the pre-image point by a scale factor of one-half toward the point (0, 10).

Q10 Write down the transformation functions for these three transformations.

Q11 Modify a copy of the fern sketch to use these transformations, with equal probabilities for all three transformations. What figure results?

Q12 Build a similar iterated function system with four transformations, dilating the pre-image by a factor of one-third toward one of the points (0, 0), (1, 0), (0, 1), (1, 1) with equal probability. Describe the resulting figure.

Q13 Research Barnsley's method and fractal compression. Report your findings.

BARNSLEY'S FERN

Objective: Students plot points using four pairs of iterated functions. By choosing randomly among the function pairs, they create a fractal Barnsley's Fern.

Prerequisites: Students should be familiar with the use of mathematical iteration to produce fractals.

Sketchpad Proficiency: Advanced. Students should be familiar with iteration and with the use of custom tools.

Class Time: 40 minutes

Required Sketch: Fractal Fern.gsp

Q1 Each of the four functions is associated with a fixed point:

r values	Fixed point
0.00–0.01	(0, 0)
0.01–0.08	(-0.61, 1.87)
0.08–0.15	(0.15, 0.63)
0.15–1.00	(2.66, 9.96)

Q2 The fourth function transforms a point on one leaf to the corresponding point on the next higher leaf.

Q3 The first function transforms a point anywhere on the fern into a point on the lowest portion of the stem. The second function transforms a point anywhere on the fern to the corresponding point on the lowest left leaf. The third function transforms a point anywhere on the fern to the corresponding point on the lowest right leaf.

Q4 The first fixed point is the base of the stem, and the fourth is the very tip of the fern. The second is the point on the bottom left leaf that's in the same relation to both that leaf and the entire fern, and the third is the point on the bottom right leaf that's in the same relation to that leaf and the entire fern.

Q5 If the first transformation is never used, the fern appears without a stem, and each leaf appears without its stem. The first transformation takes the point to the lowest part of the stem, where the 85% transformation can move it up the stem. Thus the entire stem is produced.

Q6 The second transformation takes a point to the corresponding part of the lowest left leaf. Subsequent 85% transformations generate the other left-hand leaves. Similarly, the third transformation is responsible for generating the right-hand leaves. Finally, the 85% transformation generates higher leaves from lower leaves, moving toward the tip of the fern.

EXPLORE MORE

Q7 The 0.85 in function 4 makes each succeeding leaf 85% of the size of the previous leaf. The .44 in function 3 and the 1.6 in function 2 determine the height at which the leaves begin on the two sides. The numbers clustering around .20 in functions 2 and 3 make the side leaves about 1/5 the size of the main leaf.

Q8 To make the leaves opposite instead of alternating, use the same value for *f* in the second and third functions.

Q9 Answers, and resulting shapes, will vary. You may want to ask students to print their most interesting shapes to share with the class.

Q10 The three functions use these coefficients:

0.50	0.00	3.00
0.00	0.50	0.00
0.50	0.00	-3.00
0.00	0.50	0.00
0.50	0.00	0.00
0.00	0.50	5.00

Q11 These functions produce the Sierpiński triangle.

Q12 The four functions use the coefficients

0.33	0.00	0.00
0.00	0.33	0.00
0.33	0.00	0.67
0.00	0.33	0.00
0.33	0.00	0.00
0.00	0.33	0.67
0.33	0.00	0.67
0.00	0.33	0.67

The resulting figure is a Sierpiński square: