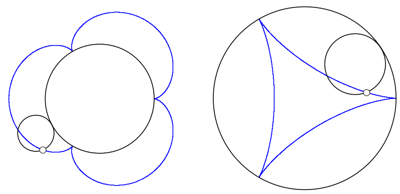
When a moving circle rolls around a stationary circle, a point on the moving circle traces an interesting curve related to the cycloid. If it rolls around the outside, the curve an *epicycloid*. If it rolls on the inside, the curve is a *hypocycloid*.



1. Open the Geometric page of **Epicycloid.gsp** in the folder **Supplemental Activities | Epicycloid**.

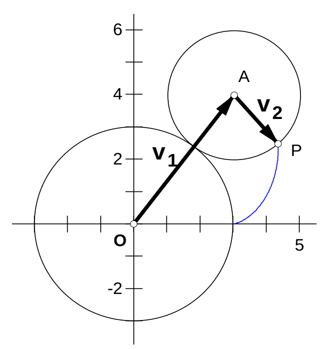
This page contains a completed geometric construction of an epicycloid. The value r_1 is the radius of the stationary circle, r_2 is the radius of the moving circle, and *cycles* controls the number of times the moving circle goes around the stationary circle before returning to the home position. Edit these values to check your answers to the questions below.

Q1 What conditions must be true in order to form a closed curve in only one cycle?

Q2 What is a general rule for determining how many cycles are needed to close the curve?

Q3 What happens when a negative value is entered for r_2?

In the rest of this activity, you will define parametric functions to describe an epicycloids or a hypocycloid. You will center the stationary circle on the origin, start the moving circle on the right, and start the traced point at the point of tangency.



To determine the parametric functions, you will add two vectors: $\boldsymbol{v_1}$ from the origin to the center of the moving circle, and $\boldsymbol{v_2}$ from the center of the moving circle to the point being traced.

2. Open the Parametric page of **Epicycloid.gsp**.

There are two pairs of parametric functions: $x_1(\theta)$ and $y_1(\theta)$ to represent $\boldsymbol{v_1}$ and $x_2(\theta)$ and $y_2(\theta)$ to represent $\boldsymbol{v_2}$. All four functions are defined as zero for now. A third pair of functions, $x(\theta)$ and $y(\theta)$, is the sum of the other two. You will use measurement θ as the parameter, with its value ranging from 0 to 2π times the number of cycles.

The functions $x_1(\theta)$ and $y_1(\theta)$ represent the components of $\boldsymbol{v_1}$.

Q4 What are the coordinates of center point A in the following image? Express your answer in terms of r_1, r_2, and θ. Use your answer to define $x_1(\theta)$ and $y_1(\theta)$.

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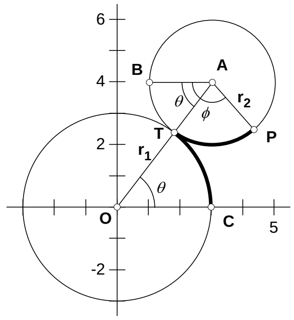


Radius AB is horizontal, and represents the position of radius AP when the motion started. With respect to the center of the moving circle, point P has moved through an angle of ϕ, the measure of $\angle BAP$. To define the parametric functions for $\mathbf{v_2}$ (from A to P), you need to know this angle in terms of θ.

Q5 Explain why the measure of $\angle BAT$ is equal to θ.

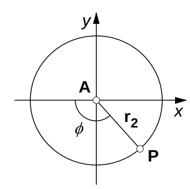
Recall this convenient fact about radian angle measurement:

(arc length) = (radius)(central angle)

Q6 Using this formula for arc length, express the length of $\overset{\frown}{TC}$ in terms of θ and r_1, and the length of $\overset{\frown}{TP}$ in terms of m$\angle TAP$ and r_2.

Q7 Notice that $\overset{\frown}{TC}$ and $\overset{\frown}{TP}$ must be equal in length. Use that fact to express m$\angle TAP$ in terms of θ, r_1, and r_2. What is ϕ in terms of θ, r_1, and r_2?

To define vector $\mathbf{v_2}$ (from A to P), consider a coordinate system with its origin at point A.



Q8 In this coordinate system, what is the x-coordinate of point P in terms of ϕ and r_2? Use your answer to Q7 to express this result in terms of θ, r_1, and r_2. Similarly, write an expression for the y-coordinate of P.

3. Edit the function definitions for $x_2(\theta)$ and $y_2(\theta)$ to match the expressions in Q8.

The calculations appear as *x(theta)* and *y(theta)*.

4. Calculate the values of $x(\theta)$ and $y(\theta)$. Plot these coordinates and label the plotted point P. Press the *Animate* button and observe the motion of the plotted point.

PRESENT

The driver point is originally hidden. Use the *Show driver* button to show it.

5. Draw the epicycloid by tracing the plotted point while the animation continues.
6. To construct the epicycloid as a locus, select the driver point and the plotted point. Then choose **Construct | Locus**.
7. To complete the picture, construct a circle with radius r_1, centered on the origin. Construct another with radius r_2, centered on point A. (You will have to plot point A. Remember that $x_1(\theta)$ and $y_1(\theta)$ represent the coordinates of A.)

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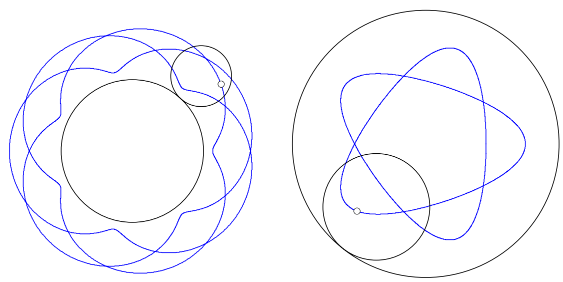
8. Using the text tool, present the definition of the curve as a single pair of parametric functions.

EXPLORE MORE

A related curve is the *epitrochoid*. In this case, the traced point moves with the moving circle, but instead of being on the circumference of the circle, it is either inside or outside of the circle. When the moving circle is inside the stationary one, the curve is called a *hypotrochoid*. Try to construct these curves by editing the sketch that you just completed. Examples of each are shown below.

The curves generated by the educational toy Spirograph fall into these families.



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EPICYCLOIDS AND HYPOCYCLOIDS

Objective: Students apply the concept of vector addition to define parametric functions for epicycloids and hypocycloids.

Prerequisites: Students should have a good understanding of parametric functions, radian angle measurement, elementary trigonometry, and vector addition.

Sketchpad Proficiency: Intermediate. A partially completed sketch has been provided in order to save time. Students edit functions and perform some simple constructions, such as plotting points and constructing a locus.

Class Time: 40–50 minutes for the basic construction and presentation. Some students will be able to pursue the Explore More section.

Required Sketch: Epicycloid.gsp

Example Sketch: Epicycloid Work.gsp

Although the construction of the epicycloid is very similar to that of the cycloid, the calculations are considerably more challenging.

Students first examine a completed geometric epicycloid construction. This gives them a clear picture of the curve before beginning their own work.

Q1 The epicycloids will close in one cycle only if r_2 divides r_1.

Q2 The number of cycles needed to close the curve is $LCM(r_1,r_2)/r_1$.

If the radii are not commensurable, the curve will not close for any finite number of cycles.

Q3 If r_2 is negative, the curve becomes a hypocycloid.

Q4 The coordinates of center point A are

$x_1(\theta) = (r_1 + r_2)\cos\theta$

$y_1(\theta) = (r_1 + r_2)\sin\theta$

Q5 Various explanations are possible. One explanation is based on the fact that segments $\overline{OC}$ and $\overline{AB}$ are parallel, and that θ and $\angle BAT$ are alternate interior angles formed by transversal $\overline{OA}$.

Q6 $m\overset{\frown}{TC} = r_1\theta$, and $m\overset{\frown}{TP} = r_2 m\angle TAP$

Q7 $r_2 m\angle TAP = r_1\theta$

$$m\angle TAP = \frac{r_1}{r_2}\theta$$

$$\phi = \theta + m\angle TAP = \frac{r_1 + r_2}{r_2}\theta$$

Q8 Be careful here, because the angle is measured from the negative x-axis, not the positive.

$$x_2(\theta) = -r_2\cos\phi = -r_2\cos\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

$$y_2(\theta) = -r_2\sin\phi = -r_2\sin\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

PRESENT

The presentation should include a clear statement of the compound parametric functions:

$$x(\theta) = (r_1 + r_2)\cos\theta - r_2\cos\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

$$y(\theta) = (r_1 + r_2)\sin\theta - r_2\sin\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

EXPLORE MORE

There are several approaches to construct an epitrochoid. One solution is to create a parameter or a slider controlling a ratio. Multiply functions x_2 and y_2 by the ratio. In the parametric functions below, k is the new ratio.

$$x(\theta) = (r_1 + r_2)\cos\theta - kr_2\cos\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

$$y(\theta) = (r_1 + r_2)\sin\theta - kr_2\sin\left(\frac{r_1 + r_2}{r_2}\theta\right)$$

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