TRANSLATION OF FUNCTIONS

CHALLENGE QUESTION (Q14)

This activity note describes in detail the Challenge question (Q14) from the Translation of Functions activity in *Exploring Precalculus with The Geometer's Sketchpad*.

You may want to suggest that students explore this challenge by investigating periodic, linear, and quadratic functions. (Complete constructions for these three cases are provided in the **Translate Work.gsp** sketch.)

Some students may realize that they can state the question in another way: Can you find a function $f(x)$ and a non-zero vector V such that the translated image is identical to the pre-image function? (If such a non-zero identity translation exists for $f(x)$, then any image of $f(x)$ translated by an arbitrary vector W will be identical to an image translated by $W + V$.)

Linear Functions

For linear functions, it's easy to find such a vector V.

$f(x) = mx + b$ (pre-image function)

$g(x) = f(x - x_v) + y_v$ (translated image)

$g(x) = m(x - x_v) + b + y_v$ (substitution)

We want to know when $f(x) = g(x)$.

$mx + b = m(x - x_v) + b + y_v$

$0 = -m\, x_v + y_v$

$y_v/x_v = m$

So any translation by vector V:(x_v, y_v) for which $y_v/x_v = m$ translates a linear function to itself. As a consequence, given an arbitrary translated image of a linear function, combining the given translation vector with such a vector V will give a different translation vector producing the same image.

Periodic Functions

Similarly, it's easy to show that you can translate any periodic function with period p horizontally by any integer multiple of p to produce an image identical to the pre-image. So you can generate any given translated image of a periodic function by an infinite number of possible translations.

Polynomial Functions

No polynomial function of degree greater than one can have two different translations that produce the same image. It's an interesting exercise to apply the same analysis used above in the linear case and see why the result requires that the translation vector be zero.

Other Functions

For linear functions, the identity translation is parallel to the slope; for periodic functions, the identity translation is horizontal. Are there other classes? For instance, are there any non-linear functions that have a non-horizontal identity translation?

We can prove that such a function exists. The function has to satisfy the following property for some non-zero V:

$$f(x) = f(x - x_v) + y_v.$$

You can construct such a function by adding a periodic function with period p to a linear function with slope m. Then a translation by a vector the (np, mnp), where n is an integer, is an identity translation.

Proof:

Let $f(x) = g(x) + h(x)$, where $g(x)$ is linear with slope m and $h(x)$ is periodic with period p.

Because $h(x)$ is periodic, we know that $h(x) = h(x + np)$.

Because $g(x)$ is linear, we know that $g(x) = g(x - np) + mnp$.

Let $t(x)$ be the translation of $f(x)$ by the vector (np, mnp). You can write this translation as

$t(x) = f(x - np) + mnp$

$= g(x - np) + h(x - np) + mnp$

$= \{g(x - np) + mnp\} + h(x - np)$

$= g(x) + h(x)$

$= f(x)$

Thus the translation by (np, mnp) is an identity translation.

Presentation Sketch

Each page from 5 to 8 of the **Translate Work.gsp** sketch contains a plot of an original function, two geometrically translated images, and a *Show Test Functions* button. Moving points V and W allows the

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student to explore the possibility of matching translated images with two different translation vectors. Press the Show/Hide button in order to display the equations and plots of algebraically transformed images. If it is possible to create the same plot of the function by two different translations, matching geometrically transformed images with corresponding algebraic images allows you to determine two different equations for the translated image of the original function. On page 8 you will find an example of the non-linear function that you can translate in two different ways to produce the same image.

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