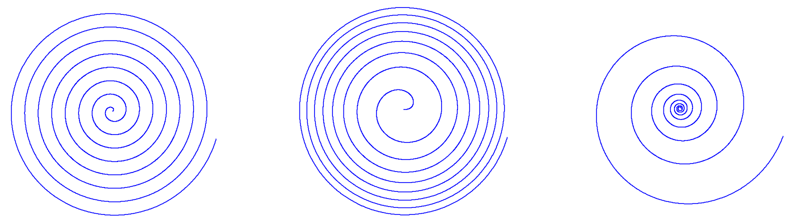
Spirals are common forms in art and nature. The word *spiral* is a general term used to describe a curved path winding away from (or toward) a point. Spirals fall into several different classes. In this activity you will explore three of those classes.



CONTINUOUS SPIRALS

1. Open **Spirals.gsp** in the folder **Supplemental Activities | Spirals**.

The Continuous page shows *Archimedes's Spiral,* defined by the polar function $r(\theta) = a\theta$. Adjacent turns of this spiral have the same separation throughout. This is the form of the groove in a phonograph record, or the cross-section of a rolled carpet.

2. Change the value of the *upper limit* parameter by selecting it and either animating it or pressing the **+** or **–** key.

Q1 What is the effect of increasing or decreasing the value of the *upper limit* parameter? What is the effect of changing the *lower limit* parameter?

3. Drag slider a and observe its effect on the spiral.

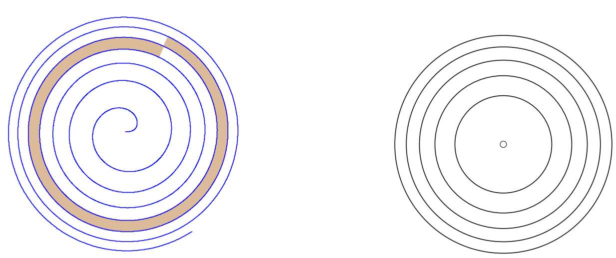
Q2 In terms of a, what is the spacing between consecutive turns of the spiral?

4. Change the function definition:

$$r(\theta) = a\sqrt{\theta}$$

To change the function definition, use the **Arrow** tool to double-click the function. Then use the Calculator that appears to edit the function. To insert the square root, position the cursor immediately before the θ and choose **sqrt** from the Calculator's Functions popup menu.

The result is *Fermat's Spiral*. In this spiral, the separation between adjacent turns diminishes as the curve moves outward. The area of the region enclosed by two adjacent turns (such as the shaded region below on the left) is constant. To understand this property, consider a simpler analogy.



An *annular region*, also called an *annulus,* is a ring—the area between two concentric circles.

Q3 Consider concentric circles with radii $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \sqrt{5}$. Find the areas of the five circles. What are the areas of the four annular regions between adjacent circles?

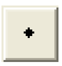
Use the **^** key in the Calculator's keypad to insert an exponent into the function.

5. Change the function once more:

$$r(\theta) = ab^{\theta}$$

This is a *logarithmic spiral*. The shape occurs in certain plants and mollusks. It is especially evident in the cross-section of a nautilus shell. Notice that this spiral, as it appears on the screen, does not reach the center point.

Q4 Adjust the b slider so that $1 < b < 1.2$. Change the parameter *lower limit* to -20. Does the spiral reach the center point now? If the limit is lowered further, is there any limit to the number of turns the curve makes as it winds inward?



6. Use the **Point** tool to construct point P on the spiral's locus. Mark the center of the spiral and dilate point P by a scale factor of $b^{2\pi}$.

Q5 Drag point P along the spiral. What is the locus of its dilated image? What property does this imply?

DISCRETE SPIRALS

Now consider a discrete set of objects lying on a spiral. This is the form spirals often take in nature.

Use the tabs at the bottom of the window to go from page to page.

7. Go to the Discrete page of **Spirals.gsp**.

In this sketch the polar coordinates of the blue dots are defined parametrically by the functions *theta*(x) and $r(x)$. A separate function, *dot scale*(x), controls the size of the dots. Parameter x_0 is the function argument for the first dot, and *depth* determines the number of dots in the iteration, creating a sequence of integers x_0, $x_0+1,\ldots,$ $x_0+depth$.

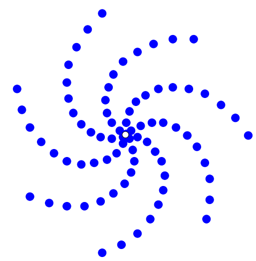
When you open the page, it shows an Archimedes spiral. Note that although these are parametric equations, r is still in direct variation with θ.

$$theta(x) = ax \qquad r(x) = bx \qquad dot\ scale(x) = c$$

To set the parameter, use the **Arrow** tool to double-click it, and then type the new value. You can also change the value by pressing the **+** or **–** key on the keyboard.

8. Interesting patterns appear in Archimedes spirals. Set the parameter *depth* to zero, and set slider a to 0.89. Select *depth*, and press and hold the **+** key.

The dots progress along the spiral, moving outward in a counterclockwise direction. Depending on the value of a, when the depth is greater than 50, other swirls begin to appear, perhaps moving in the opposite direction.

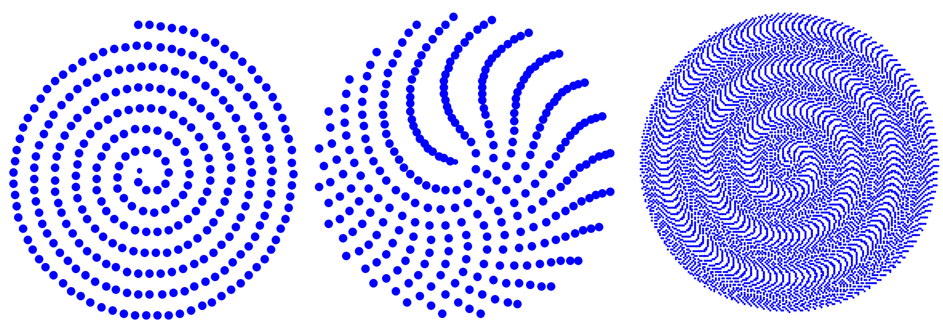


Q6 Are the secondary swirls also spirals? Are they Archimedes spirals?

9. It is difficult to follow the main spiral because consecutive dots grow further apart as the path diverges. To make the distance between consecutive points constant, while maintaining a direct variation, apply these changes:

$$theta(x) = a\sqrt{x} \qquad r(x) = b\sqrt{x} \qquad dot\ scale(x) = c$$

Now the dots are like pearls on a string. Experiment with changing the sliders. Many interesting patterns appear when *b* and *c* are very small, and *depth* is very large. The image on the right has 10,000 dots.



10. Next, create Fermat's spiral. Use these function definitions and watch the patterns as the sliders change.

$$theta(x) = ax \qquad r(x) = b\sqrt{x} \qquad dot\ scale = c$$

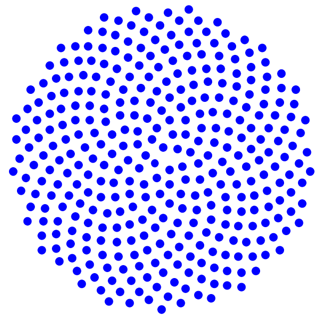
11. Here is another nature connection. Create a calculation for ϕ (*phi*), the golden ratio. Then change the definition of *theta* as shown:

$$phi = \frac{1+\sqrt{5}}{2} \qquad theta(x) = \frac{2\pi}{phi}x$$

Q7 Adjust *c* so that you can easily distinguish individual dots. Where in nature might this pattern be observed?

This pattern uses space efficiently. All parts of the plane have the same density of dots, a property that may help explain why the pattern tends to show up in nature.

Q8 The pattern below is a Fermat spiral. Count the left-hand swirls and the right-hand swirls. Do you see any special significance in the results?

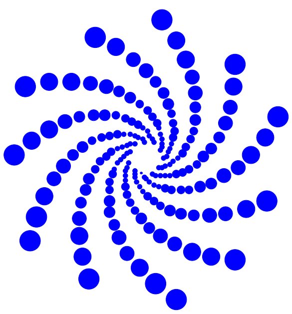


12. The logarithmic spiral has a way of quickly growing out of control. To prevent overloading your computer, first set *depth* to 50 or less and make c a very small positive number. Now use these settings for the logarithmic spiral:

To make c small, drag the slider's white unit point as far right as possible. Then drag the index point near the left end of the slider.

$$theta(x) = ax \qquad r(x) = b(1+c)^x \qquad dot\ scale(x) = d(1+c)^x$$

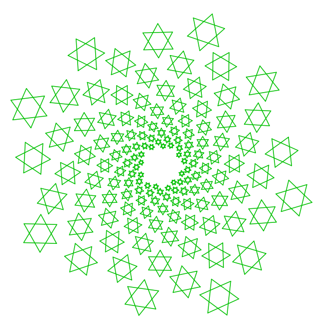
After adjusting the sliders to create a good visual pattern, raise the value of *depth*. Notice that the *dot scale* function is now variable. Unlike the previous curve, these dots grow further apart as the spiral winds outward. We fill the space by having the dots grow at the same rate.



Q9 Certain settings of the slider a cause the appearance of secondary swirls. Describe their paths. What happens if $q(x) = (2\pi/phi)x$?

EXPLORE MORE

Spirals appear extensively in art — in painting, pottery, architecture, graphic design, and so forth. Go to the Custom page of **Spirals.gsp**, and follow the instructions for mapping a small design onto a spiral. Experiment with changes to the functions.



SPIRALS

Objective: Students learn properties of three kinds of spirals, presented as continuous curves and as discrete objects on those curves.

Prerequisites: The logarithmic spirals require some understanding of exponential functions, and there is a connection with Fibonacci numbers. Otherwise the activity is mostly a matter of making observations in the provided sketches.

Sketchpad Proficiency: Beginner. Students edit functions and operate slider controls. The extension is more advanced, requiring students to use a Custom tool and create an iteration.

Class Time: 40 minutes. The extension is open-ended, and could take much longer. This activity could be done in parts as a presentation.

Required Sketch: Spirals.gsp

CONTINUOUS SPIRALS

Q1 Increasing the upper limit extends the outer end of the spiral, so that the spiral goes farther out from the center. Increasing the lower limit to values greater than zero moves the inner end of the spiral out so it does not touch the center point. Decreasing the lower limit to values less than zero starts a second spiral, opposite in direction to the first.

Q2 Consecutive turns of the spiral are separated by a distance of $2\pi a$.

$r(\theta+2\pi) - r(\theta) = 2\pi a$

Q3 The circle areas are π, 2π, 3π, 4π, and 5π. The annuli formed by adjacent circles each have an area of π.

Q4 The spiral will not reach the center for any finite number of turns. For positive a and b, the radius ab^x is positive for any real number x. There is no limit to the number of turns the spiral makes as it gets closer and closer to the center.

There is a limit to the length of the spiral as it winds inward, but a typical precalculus student could only guess at that concept. This is the limit of the length of the part of the spiral below $x=0$:

$$\frac{a\sqrt{1+(\ln b)^2}}{\ln b}$$

Q5 The image of point P will always fall on the next turn of the spiral. This indicates that the logarithmic spiral is self-similar.

DISCRETE SPIRALS

Q6 The secondary swirls are Archimedes spirals.

Q7 Answers vary. Discrete Archimedes spirals appear in pine cones, sunflowers, and pineapples.

Q8 There are 34 left-hand swirls and 21 right-hand swirls. These are Fibonacci numbers. As the depth is increased, the swirls (which are Fermat spirals) tend to a circular path, and the eye tends to pick up a different series of swirls. For this reason, the swirl counts may vary, but they should always be Fibonacci numbers.

Students can count the swirls on the computer screen or on paper, but counting swirls is much easier on paper.

Q9 The secondary swirls are also logarithmic spirals. When $\theta(x) = (2\pi/phi)x$, Fibonacci numbers can be found in the numbers of swirls, just as they were found in the Fermat spiral.

EXPLORE MORE

When the Custom tool **Map** is used to map a design to the spiral, all of the colors on the image will be black. Change them before executing the iteration.

Encourage experimentation. There are other spirals to be discovered. Besides making creative images, students might add some animation to the image. It is possible to make each of the individual objects rotate about its own center, or to make the entire spiral rotate.