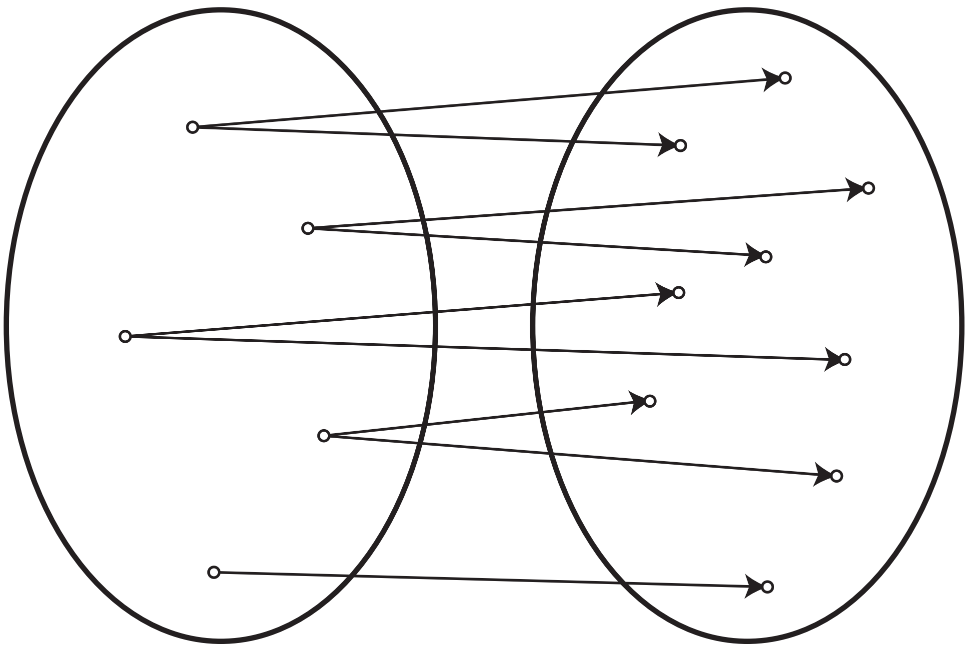
2

Functions and Relations

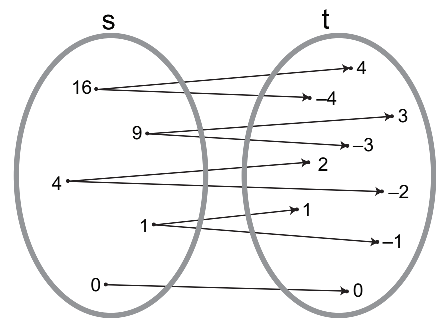


Relations and Functions

A *mathematical relation* exists when two mathematical values are related in some way. A *function* is a specific kind of relation. In this activity you'll explore several relations, decide which of them are functions, and develop a graphical test to determine whether a relation is a function.

EXPLORE

Q1 Think about the mathematical relation between your location and the time. Is it possible for you to be in two different places at the same time? Is it possible for you to be in the same place at two different times? Explain your answers.



Understanding the answers to these two questions will help you to differentiate between relations that are functions and those that are not.

1. To explore a relation between position and time, open **Relations and Functions.gsp.**
2. Press *Run!* to see Flopsy steal a carrot. Notice the variables *s* (which shows Flopsy's position) and *t* (which shows the time). When Flopsy finishes running, press *Reset.*
3. Create a table to record the position and time. Double-click the table to make the first row of values permanent.

To create the table, select measurements *s* and *t* in order, and choose **Graph | Tabulate.**

4. Press *Dart!* to make Flopsy dart forward. Double-click the table to record her new position and time.
5. Continue moving Flopsy (by pressing *Dart!*) and recording the data (by double-clicking the table) until Flopsy has returned to her original position. The table should contain nine rows of permanent data when Flopsy finishes.

Definition: A *relation* is any set of ordered pairs, such as the ordered pairs (s, t) that appear in your table.

Q2 Review the data in this relation involving *s* and *t*. Is it possible for Flopsy to be in two different places at the same time? Is it possible for her to be in the same place at two different times? Explain by referring to the data.

6. Page 2 of the sketch shows the same table for *s* and *t*, and also shows two large bubbles containing all the numbers that appear in the table.

Q3 Press the *When was she at s = 4?* button. Answer this question using the arrows that appear. How does this relate to the questions that opened this activity?

7. These two arrows show part of the mapping from position (s) to time (t). Reset the maps and then use either Show button to show the full mapping.

Q4 From the arrows, are there any ordered pairs that have the same first element (s) but different second elements (t)? Which ones?

8. Go to page 3. You will look at the same data, in the form (t, s) rather than (s, t).

Q5 Press the *Where was she at t = 2?* button. Where was Flopsy when the time measurement was 2?

9. The arrows on this page show the mapping from time (t) to position (s). Reset the maps and then use either Show button to show the full mapping.

Q6 From the arrows, are there any ordered pairs that have the same first element (t), but different second elements (s)? Which ones?

When a relation is a function, the first element is called the *independent variable,* and the second is the *dependent variable*.

Definition: A *function* is a relation for which there is exactly one second element for each first element. (If t is the first element and s is the second element, this means that Flopsy cannot be in two different places at the same time.)

Q7 Is the relation (t, s) shown on page 3 a function? What about the relation (s, t) shown on page 2? How can you tell from the arrows?

EXPLORE MORE

You will now develop a visual test for determining whether a relation is a function.

Q8 Page 4 contains two movable points that are part of a relation. Try to drag the two points so that they both have the same first element but different second elements. Can you do this? Is the relation a function? If not, what values did you use to prove it's not? Where are the points located relative to each other?

Q9 Press the *Problem b* button and try dragging these points. Can you make both of them have the same first element but different second elements? If so, what are the values, and where are the points relative to each other? Is it a function? Also try problems c and d on this page.

Q10 Page 5 shows the graph of a relation, and also shows a movable vertical line. Drag the line back and forth to be sure it remains vertical. How can you use the vertical line to tell whether the graph represents a function?

Q11 Investigate relations b, c, and d on page 5. Which are functions? For each one that's not, at what coordinate did you place the vertical line to prove that it's not?

Objective: Students explore the definitions of relation and function using a model involving position and time, analyze one-to-many and many-to-one mappings, and develop a vertical line test for functions.

Student Audience: Algebra 1/Algebra 2

Prerequisites: It may be helpful (but not critical) if students have already had a brief introduction to relations and functions. The activity includes the definitions of the terms, as well as multiple examples.

Sketchpad Level: Easy. There's very little actual construction to do.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Relations and Functions.gsp**) or Whole-Class Presentation (use **Relations Functions Present.gsp**)

EXPLORE

Use this activity when introducing students to the concepts of relations and functions. The example of the rabbit is memorable and will remind students of their conclusion that time cannot be a function of location.

Q1 It is not possible to be in two places at the same time. It is possible to be in the same place at two different times. Explanations will vary, but this can generate an interesting discussion. Teachers generally describe the independent variable as a variable that you have control over—one that you have the ability to vary. But for these measurements, the dependent variable (location) is the one you can control, whereas the independent variable (time) proceeds rudely onward, oblivious to any attempt to control it.

Q2 Flopsy cannot be in two locations at the same time, but she is at position 9 (for instance) at both $t = 1$ and $t = 7$. Thus she's at the same place at two different times.

Q3 Both arrows lead away from 4 in the position bubble, leading to 2 and to 6 in the time bubble. Thus Flopsy was at this position at two different times: two seconds after starting and six seconds after starting.

Q4 Several values of position correspond to more than one ordered pair (in other words, to more than one row of the table). There are two pairs with a first element of 16, two with a first element of 9, two with a first element of 4, and two with a first element of 1.

Q5 When $t = 2$, Flopsy was at position 4.

Q6 There are no ordered pairs with the same time and different locations—that is, none with the same first element and a different second element. This is the fundamental definition of a function.

Q7 The relation (t, s) on page 3 is a function. The relation (s, t) on page 2 is not. You can tell from the arrows because there's only a single arrow leading from any given first element.

EXPLORE MORE

Q8 You can drag the points so that they have the same first element and different second elements. For instance, the ordered pairs (3.50, 1.55) and (3.50, 6.72) are both part of the relation. Therefore, this relation is not a function. These points are aligned with each other vertically.

Q9 Problem b is a function; there are no ordered pairs with the same first element and different second elements. Problem c is also a function. Problem d is not a function; numeric data to demonstrate this will vary, but students should observe that they are able to arrange the two points so they are aligned vertically.

Q10 This is not a function, because there are several places where the same first element corresponds to more than one second element. The vertical line demonstrates this condition by intersecting the function plot in two or more places.

Q11 Problem b is a function, and the vertical line can never intersect it at more than a single place. Problem c is the inverse of the cosine function; this inverse is not a function itself, because the vertical line intersects it in many places. The inverse cosine must be defined to have a restricted range if it is to be dealt with as a function. Problem d is a step function. Though the vertical line comes close to two different segments of the graph at certain positions, in fact it never intersects the two segments at the same time.

The presentation should follow the sequence in the student activity. Begin with an informal description of relations and functions, and use the presentation to motivate a more precise definition.

Start with these descriptions: A *mathematical relation* exists when two mathematical values are related in some way. A *function* is a specific kind of relation.

EXPLORE

Q1 Can you be in two different places at the same time? Can you be in the same place at two different times? Ask a number of students to respond and explain.

1. Open **Relations and Functions Present.gsp** and press *Run!* to see Flopsy steal a carrot. Run the animation again and point out the variables s (which shows Flopsy's position) and t (which shows the time).
2. Reset Flopsy. To record her position and time, press *Show Table* and double-click the table to make the first row of values permanent.
3. Press *Dart!* repeatedly to make Flopsy dart forward. At the end of each movement, double-click the table to record the new position and time. You should have nine rows of permanent data when you finish.

This table does not define a function. Our purpose is to clarify the distinction, so the first relation students work with should not be a function.

Give students a more precise definition, now that it's illustrated by a table: A *relation* is any set of ordered pairs, such as the ordered pairs (s, t) that appear in your table.

6. On page 2, press the *When was she at* $s = 4$*?* button. Question students, and discuss the meaning of the arrows, until they are all convinced that there's not just one right answer to this question.
7. These two arrows show part of the mapping from position (s) to time (t). Reset the maps and then use either Show button to show the full mapping.

For a relation to be a *function,* each input value must correspond to only a single output value.

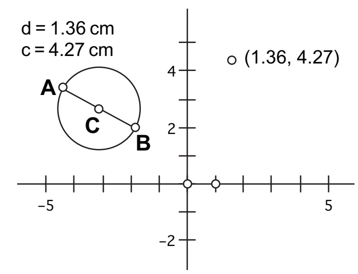
8. Contrast the full mappings (with all arrows showing) from pages 2 and 3 to further clarify the distinction between the relation that is a function (page 3) and the one that is not (page 2).
9. Use page 4 to see which problems allow ordered pairs that violate the function definition and to point out that the two ordered pairs are aligned vertically when they prove a relation is not a function.
10. Use page 5 to demonstrate the vertical line test for functions.

Finish with a class discussion reviewing the definitions. Ask students why they think functions might be so important in mathematics. (One benefit of working with functions is that the ability to find a unique output, given any input, makes many problems much easier to manage.)

The Circumference Function

In a *function,* one quantity depends on another quantity, just as the number of songs a jukebox plays depends on how much money is put in. In the case of a circle, we can say that its circumference *depends* on its diameter—the farther it is across a circle, the farther it is around it. In this activity you'll explore this connection as a *functional* relationship between two changing quantities.

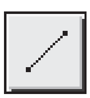
IMAGINE AND PREDICT



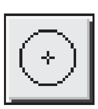
Imagine using a compass to draw a circle. You measure the circle's diameter (1.36 cm) and calculate its circumference (4.27 cm). You then plot the point (1.36, 4.27) on a piece of graph paper. You draw a second circle, measure its diameter (d), calculate its circumference (c), and plot a second point (d, c). You do this for many more circles.

Q1 What will your graph look like after you plot many points? Discuss this with your group. Will the graph be straight or curved? Why? If it's straight, what will its slope be? If it's curved, will it bend up or down? Will it go forever in both directions, or will it start or stop somewhere? Write down your predictions.

SKETCH AND INVESTIGATE

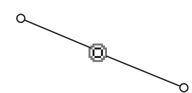


The **Segment** tool

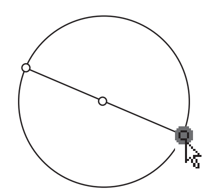


The **Compass** tool

Now test your prediction using Sketchpad.



1. In a new sketch, construct a segment using the **Segment** tool. With the segment selected, choose **Construct | Midpoint.**



2. Construct a circle whose diameter is the segment. To do this, choose the **Compass** tool and click first on the midpoint and then on one endpoint. Drag each endpoint of the diameter to make sure your construction holds together.
3. Measure the segment's length by selecting it and choosing **Measure | Length.** Similarly, select the circle and measure its circumference.
4. Label the diameter measurement d and the circumference measurement c.
5. Plot the point (d, c) by selecting the measurements in order and choosing **Graph | Plot As (x, y).** The plotted point appears.
6. Select the plotted point and choose **Display | Trace Plotted Point.**

Double-click a measurement with the **Text** tool (the *A* in the Toolbox) to edit its label.

You can drag the tick mark numbers on the axes to change the scale of the graph.

Q2 The moment of truth has arrived. Drag either endpoint and observe the point being traced. Describe this trace on your paper. Was your prediction correct?

FURTHER INVESTIGATION

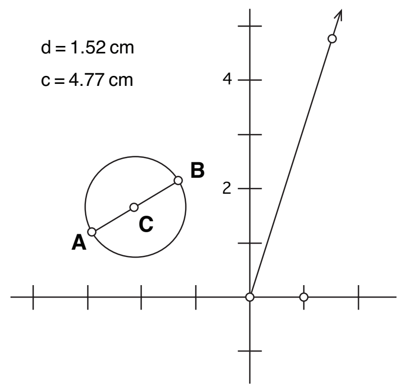
You've now seen the shape of the diameter-circumference trace. But why does it look the way it does? You may know that lines can be modeled with equations of the form $y = mx + b$, or $c = md + b$ in this case. So what are m and b for this line?

Q3 Why does it make sense that the trace goes through the origin? What does this tell you about m and b in the equation $c = md + b$?

Q4 Drag the diameter endpoints so that d is as close to 1.00 as you can make it. What is the approximate circumference of the circle? Is this number familiar to you? If so, what's it called?

Choose **Measure | Calculate** to open the Calculator. Click on *c* and *d* in the sketch to enter them into the calculation.

Q5 Use Sketchpad's Calculator to find the ratio c/d. Drag one of the diameter endpoints and observe the ratio. What happens and why?



7. Turn off tracing for your plotted point.

8. Construct a ray from the origin through the plotted point. Then measure the ray's slope.

Q6 What does the slope measurement tell you about m and b in $c = md + b$?

You've now seen the same number in three places:

- the circumference of a circle with diameter 1
- the ratio of any circle's circumference to its diameter
- the slope of the graph of a circle's circumference as a function of its diameter

Choose **Graph | Plot New Function** and enter your proposed expression. Use *x* for the diameter (because you used *d* for the *x*-coordinate earlier).

Q7 Use your results for m and b from Q3 and Q6 to plot a line that includes the ray from step 8. On your paper, write the function you used.

EXPLORE MORE

Q8 What's an appropriate domain for the circumference function? Select the function plot from Q7 and choose **Properties** from the Edit menu. Go to the Plot panel and set an appropriate domain for this situation.

Q9 Consider a circle's *area* as a function of its radius. How will the plot of that function compare with the diameter-circumference plot? Make a prediction, then use the techniques from this activity to confirm.

Objective: Students construct a circle based on its diameter. They then measure, graph, and analyze the function that connects the diameter and circumference of a circle.

Student Audience: Pre-algebra/Algebra 1/Algebra 2

Prerequisites: Students should have had a basic introduction to functions.

Sketchpad Level: Intermediate

Activity Time: 30–40 minutes. The time will vary depending on students' Sketchpad experience, since they do all the construction steps themselves.

Setting: Paired/Individual Activity (no sketch needed) or Whole-Class Presentation (use **Circumference Fn Present.gsp**). The pages of **Circumference Fn.gsp** show the activity as it's supposed to be after steps 4, 6, Q4, Q5, and Q7.

Related Activities: Relations and Functions (to introduce the concepts), Radius and Arc Length (measuring and analyzing an existing construction), Functions in a Triangle (also involving construction), Functional Geometry (finding functions in a variety of existing constructions)

IMAGINE AND PREDICT

One of the interesting things about this activity is the relationship between *formulas* and *functions.* Most students at this level know the formula $c = \pi \cdot d$, and most can graph the line $y = 3x$ or $y = 3.14x$ or even $y = \pi x$. Yet very few are able to recognize the first formula as being equivalent to the final equation. In this activity, students are challenged to think of c as a dependent variable just like y, and of d as an independent variable just like x. Rich connections can result from a lively classroom discussion.

Q1 It is important that students take the opening part of the activity—Imagine and Predict—seriously. This gives them an investment in the activity and exercises an important part of their mathematical thinking. The key issue here isn't right or wrong answers, it's the quality of the explanations. Use your judgment about providing hints during this section; too much information may diminish some students' interest in the rest of the activity.

SKETCH AND INVESTIGATE

2. Students need to click first on the midpoint and then on one of the endpoints for the circle to be attached properly. The most common mistake students make is to click the second time in blank space in such a way that the circle may *appear* to pass through the endpoint but is not *constructed* to do so. Dragging either endpoint will reveal this problem.

3. The measurement commands won't be available if too many objects are selected. Students can deselect all objects first by clicking in blank space.

Q2 The trace is a ray, emanating from the origin. More details to come in subsequent answers!

FURTHER INVESTIGATION

Q3 It makes sense that the graph goes through the origin, because a circle with a diameter of zero would have a circumference of zero. If students don't accept that a zero diameter, zero circumference circle exists, use a limit argument: Drag an endpoint so the circle has a tiny diameter, and watch the two measurements approach zero as the plotted point approaches the origin. If the two endpoints actually coincide, the construction and measurements disappear. (This is a nice situation in which to investigate a limit and see why it's important to make the quantity as small as you can—but *not* to make it zero.)

That the trace goes through (or at least to) the origin tells us that the y-intercept (b) is 0. However, this fact tells us nothing about the slope (m).

Q4 The circumference is approximately 3.14. This number is π (to the nearest 0.01). Students can use the properties of the measurement (**Edit | Properties | Value**) to increase the precision, but the accuracy of the result still depends on how close to 1 they can make the diameter. Consider challenging students to find a method of constructing a circle with a diameter of *exactly* 1.

Q5 The ratio stays the same—a close approximation of π—no matter the size of the circle. This approximation is closer than the one from the previous question, because the diameter is incorporated into the calculation. The result doesn't depend on dragging the diameter to a precise value.

8. Students can use the **Ray** tool to construct the ray. (Press on the current **Straightedge** tool, then drag and release over the **Ray** tool in the palette of tools that appears.) Or they can use **Construct | Ray** after selecting the origin and the plotted point.

Q6 It tells us that $m = \pi$ since m represents a line's slope. The construction assumed that $b = 0$, because the ray started from the origin.

Q7 The function students should plot is $f(x) = \pi \cdot x$.

EXPLORE MORE

Q8 The appropriate domain is either $x > 0$ or $x \geq 0$. (This distinction could make for an interesting discussion!)

Q9 The plot will be a parabola. Specifically, it will be the parabola with the equation $y = \pi x^2$, which corresponds with the familiar area formula $A = \pi r^2$. It has the same domain as the linear graph.

WHOLE-CLASS PRESENTATION

Use **Circumference Fn Present.gsp** to present this activity to the whole class. Use the animations provided in that sketch to view the behavior and measurements, answer questions, and stimulate a class discussion.

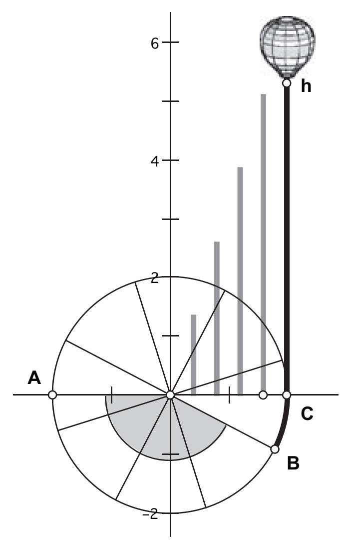
Radius and Arc Length

In this activity you'll explore the relationship between the radius and the arc length of a semicircle, and decide whether this relationship is a function.

Suppose you want to build a moving sculpture for your sister's birthday. The sculpture will be made of eight wheels of different sizes, with a string attached to each wheel. When the sculpture is in motion, a balloon attached to the other end of each string flies upward, pulling the string and spinning the wheel one-half of a revolution. To properly design the sculpture, you need to know how the height each balloon will reach is related to the radius of its wheel.

RADIUS-ARC LENGTH RELATIONSHIP

1. Open **Radius and Arc Length.gsp.** Press *Go Up* to watch a single wheel and balloon in action.
2. The actual sculpture will have eight wheels of different radii. Set a different radius by pressing *Go Down* and then dragging point *C*.
3. Operate this different-size wheel by pressing *Go Up*.



Four traces are finished and a fifth trace is in progress.

Q1 What do you observe?

4. Repeat steps 2 and 3 six more times. Try to space the wheel sizes fairly evenly.

You can erase the traces at any time by choosing **Display | Erase Traces.**

You should have eight traces left showing where the balloon went.

Q2 Describe how the height reached by the balloon changes as the radius of the circle changes.

Q3 If you draw a line through the tips of the traces, will it be straight or curved?

Next, you will collect some data on the circle radius and string length.

5. Erase the traces by choosing **Display | Erase Traces.**
6. With the balloon in its upper position, measure the *x*- and *y*-values of point *h*.

Select the point and choose **Measure | Abscissa (x).** Select the point again and choose **Measure | Ordinate (y).**

7. Put these measurements in a table by selecting them and choosing **Graph | Tabulate.** Enter the current values permanently by double-clicking the table.
8. Operate the balloon again with a different-size circle. When the motion stops, record the next set of coordinates by double-clicking the table.

9. Repeat step 7 until you have data from eight circles of different sizes.

10. Plot the data from the table by selecting it and choosing **Graph | Plot Table Data.** Make sure the options are correct before closing the dialog box.

Q4 How are the plotted points arranged? Do they appear to be in a straight line? What else do you observe about their arrangement?

Use the **Line** tool, or select the two points and choose **Construct | Line.**

11. Construct a line through the two plotted points that are farthest apart.

Q5 Does this line appear to go through all the other points? Check your answer by measuring the distance from several of the other points to the line. (To do this, select one point and the line, and choose **Measure | Distance.**) What do your results tell you?

Q6 Measure the equation of the line. What are the slope and intercept?

Q7 What does the value of the intercept tell you? Explain why this makes sense.

Q8 Where have you seen the value of the slope before? Explain why this number makes sense.

Q9 Is the relation between the radius and the string length a mathematical function? Why or why not?

Q10 Write the equation for the function using l for string length and r for radius.

Q11 Write equations representing the functions (r, c) and (d, c), where c is the circumference and d is the diameter. Explain how you can figure out each of these equations from the equation you wrote for Q10.

EXPLORE MORE

12. On page 2, operate the wheel by hand by dragging B. Notice the measurements that show the angle and distance through which you have moved B.

After plotting the data, adjust the scales on the x-axis and y-axis to show the data better.

Q12 Do you think there's a simple relationship between the angle and the length of the arc? Leave the radius constant, gather data in a table, and plot the table data. What do you observe about the plotted data? What is the equation of the data? Can you find an interpretation for the constant in this equation?

Your result will reveal an angle that's important in later mathematics.

Q13 Drag B so that the arc length is as close to the radius as you can make it. What's the arc angle? Tabulate the radius, arc length, and arc angle for several different values of the radius, each time making the radius and arc length as close to equal as you can. What do you observe about the arc angle?

Radius and Arc Length

Objective: Students use a Sketchpad model to explore the relationship between the radius of a circle and the arc length of a semicircle to decide whether it is a mathematical function, and if so to find a mathematical equation for it.

Student Audience: Algebra 1/Algebra 2

Prerequisites: None

Sketchpad Level: Easy to Intermediate. Students manipulate a pre-made sketch and tabulate and plot data.

Activity Time: 30-40 minutes

Setting: Paired/Individual Activity (use **Radius and Arc Length.gsp**) or Whole-Class Presentation (use **Radius and Arc Length Present.gsp**)

RADIUS-ARC LENGTH RELATIONSHIP

Q1 The second wheel made a different vertical bar, in a different horizontal position and of a different height.

Q2 The farther point C is dragged from the origin, the higher the traced bar is. In other words, the greater the radius of the wheel, the longer the string is and the higher the balloon goes.

Q3 The line appears to be straight.

Q4 The plotted points appear to be in a straight line, and it appears that the line may also go through the origin.

Q5 The constructed line does appear to go through all the points. The distance measurements will confirm this; they should all be zero.

Q6 The equation is approximately $y = 3.14159x + 0.00000$. The slope is 3.14159, and the intercept is very nearly zero. (These results will be more accurate if students use points far apart, and will be less accurate, perhaps significantly so, if they use points that are very close to each other.)

Q7 The intercept of zero indicates that the line goes through the origin. This makes sense: If the radius of the wheel were zero, the length of string would also be zero. Because the line goes through the origin, this function is an example of *direct variation.*

Q8 The value of the slope is π. This makes sense because the circumference of a circle is $2\pi r$, so the arc length of half a circle should be πr.

Q9 This relation is a mathematical function, because each possible value of the radius is associated with only a single string length.

Q10 The equation is $l = \pi r$. Using function notation, students could also write $l(r) = \pi r$.

Q11 The equations are $c = 2\pi r$ or $c(r) = 2\pi r$ and $c = \pi d$ or $c(d) = \pi d$. To figure the first equation out from the equation in Q10, multiply by 2 because the entire circumference is twice as long as the string. To figure the second equation out from the equation in Q10, you must realize that the diameter is also twice as long as the radius. Because both numbers are twice as big, the same equation works.

EXPLORE MORE

Q12 There is a simple linear relationship between the angle and the length of the arc. Like the previous example, this is direct variation. The equation is $length = k \cdot angle$, where k is a constant that depends on the radius. In fact, $k = \pi r/180$ because at 180° the arc will be a semicircle and the arc length will be equal to πr.

Q13 No matter the size of the wheel, the arc angle must be close to 57.3° to make the arc length equal to the radius. This angle is called a *radian* and is very important in later mathematics.

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Radius and Arc Length Present.gsp** to present this activity to the whole class.

In this presentation you'll gather and plot data about the radius and arc length of a semicircle, decide whether the relation is a function, and measure the equation.

Start by describing the model, as in the introduction to the student activity pages.

RADIUS-ARC LENGTH RELATIONSHIP

1. Open **Radius and Arc Length Present.gsp.** Press *Go Up* to show the wheel and balloon in action.

To change the radius, drag point *C* or press *Increase Radius* or *Decrease Radius.*

2. The actual sculpture will have eight wheels of different radii. Set a different radius by pressing *Go Down* and then increasing or decreasing the radius.
3. Operate this different-size wheel by pressing *Go Up*.

Q1 Ask students what they observe about the height.

4. Measure the height, and mark it by pressing the *Mark Height 1* button.

Double-click the table to record the current values permanently.

5. Make a table to record the data by selecting the radius and height measurements and choosing **Graph | Tabulate.**
6. Make at least four more measurements, each time recording the data in the table and marking the height of the balloon with one of the numbered points.

You can erase the traces at any time by choosing **Display | Erase Traces.**

You should have at least five traces left showing where the balloon went. You should also have at least five rows of data in your table and five of the height traces marked by numbered points.

Q2 Ask students to use the table to decide whether the relation is a function.

Q3 Ask whether a line through the tips of the traces will be straight or curved.

Use the **Line** tool or the **Construct | Line** command.

7. Construct a line through two of the height markers.

Q4 Ask students whether the line appears to go through all the data points or only some of them. Ask them to estimate the slope of the line and the y-intercept.

Select the measurement, choose **Edit | Properties,** and change the precision in the Value panel.

Q5 Measure the line's slope. Change the precision of the measurement to hundred-thousandths. Ask students whether they've seen this number before.

Q6 Ask students to write the equation of the line. Then measure the equation and compare their equations with the measured one.

Use page 2 to investigate the angle of rotation for which the arc length is equal to the radius. Follow the directions on the sketch.

You may also want to mention that students will see this angle later.

Q7 Ask students how this angle depends on the radius. What do they think about their results?

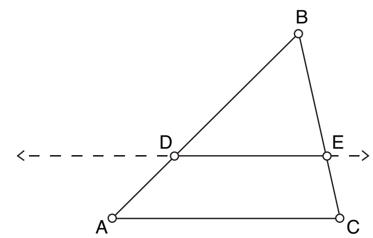
Q8 Tell students this angle is called a *radian;* ask how they think it got its name.

Functions in a Triangle

In this activity you'll construct a triangle, make two measurements, change the shape of the figure, and observe how these measurements are related. This kind of relationship—where one quantity changes as a result of another quantity changing—is an example of a function.

CONSTRUCT A TRIANGLE

Choose **Edit | Preferences,** click on the text tab, and check "For All New Points."



1. In a new sketch, change Sketchpad's preferences to label points automatically as you construct them.
2. Construct $\triangle ABC$ using the **Segment** tool.
3. Construct point D on side AB by using the **Point** tool.

Select D and $\overline{AC}$. Then choose **Construct | Parallel Line.**

4. Construct a line through point D parallel to $\overline{AC}$.
5. Construct point E where this line intersects $\overline{BC}$. To do this, click on the spot with the **Arrow** tool.
6. Hide the parallel line and construct $\overline{DE}$.
7. Measure the distance from B to $\overline{DE}$ by selecting the point and segment and choosing **Measure | Distance.** Then measure the length of $\overline{DE}$ by selecting the segment and choosing **Measure | Length.**

FUNCTIONAL RELATIONSHIPS

Now that you've measured your construction, explore the relationship between the two measurements.

8. Drag point D back and forth. Observe how the measurements change and think about how they relate to each other.

Q1 Is it possible to drag any of the points in a way that makes one measurement get larger while the other one gets smaller? What can you conclude from dragging?

The behavior of these measurements suggests that they relate to each other according to a rule. To investigate this rule and decide whether it's a mathematical function, you can graph it.

First, you must decide which of the two measurements to use as the independent variable. You can (a) make the distance from B to $\overline{DE}$ the independent variable and the length of $\overline{DE}$ the dependent variable, or (b) make $\overline{DE}$ the independent variable and B to $\overline{DE}$ the dependent variable.

Q2 Decide which of these choices (a or b) makes the most sense to you. Explain your reasoning.

Q3 On your paper, draw an *x*- and a *y*-axis, and a prediction of what you think a graph of the relationship might look like. Don't bother to scale the axes or to plot points of actual measurement values. Use only your intuitive sense of how the measurements change in drawing your prediction.

If you don't see the plotted point right away, make your triangle smaller or drag any one of the tick mark numbers on the *x*-axis toward the origin.

9. Select the two measurements—first the independent variable, then the dependent variable—and choose **Graph | Plot As (x, y).** A set of axes and a plotted point will appear. Drag point *D* and observe the path of the plotted point on the grid.
10. To see the set of all possible plotted points, select the plotted point and choose **Display | Trace Plotted Point.** Then drag *D* again, moving it all the way from *A* to *B* to see the entire graph. (You can erase the traces at any time by choosing **Display | Erase Traces.**)

Q4 How close was your prediction? What does the graph's shape tell you about the functional relationship in the triangle?

The *domain* of a function is the set of possible values for the independent variable. The *range* is the set of possible values for the dependent variable.

Q5 Describe the domain and range of this function. What do the maximum values of each correspond to in the construction?

11. To make the graph permanent, select both *D* and the plotted point, and choose **Construct | Locus.** Also turn off tracing for point *D.*
12. Drag any vertex of the triangle to change the triangle's shape and size. Observe how this affects the graph.

Q6 Describe the different types of graphs you get with different kinds of triangles. For example, what shapes give graphs with steeper slopes? Less steep slopes? Try to come up with general rules that describe how the function changes as you alter the shape of the triangle.

EXPLORE MORE

With the New Function dialog box open, click on the measurements in the sketch to enter them into the function equation.

13. Choose **Graph | Plot New Function.** Make use of the measurements in the sketch to plot a function that coincides with your locus graph.

Q7 How do you think the plots of length $\overline{BD}$ versus length $\overline{DE}$ or length $\overline{BE}$ versus length $\overline{DE}$ would compare to your existing plot? Draw a graph of your prediction on paper. Then test it by measuring the distances and plotting a point as you did before.

Q8 Make other measurements in the construction and investigate other functional relationships. See if you can find one that's not linear.

Objective: Students measure constructions in a triangle, investigate the relations that result, and plot graphs to study how they behave and whether they are functions.

Student Audience: Pre-algebra/Algebra 1/Algebra 2

Prerequisites: Some basic familiarity with graphing in the *xy* plane

Sketchpad Level: Intermediate. This is a good first activity for students learning to make their own sketches.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (no sketch needed). The pages of **Triangle Functions.gsp** show the activity at steps 5, 7, and 10.

Related Activities: The Circumference Function is similar to this activity; it also involves creating a construction from scratch and investigating relationships between various measurements. By constructing the geometric figure from scratch, students are involved more intimately in the geometric behavior and functional relationships they explore. They are more likely to remember the insights they gain about what a function is and how functions work because they have created the basis for the activity themselves.

The activity Relations and Functions is a good introduction to the difference between relations that are functions and relations that are not. The Functional Geometry activity asks students to make and analyze measurements on a variety of pre-made sketches, with a lot of emphasis on the choice of independent and dependent variables.

CONSTRUCT A TRIANGLE

The key geometric relationship in this activity is that $\triangle DBE$ is similar to $\triangle ABC$. As D travels along segment AB, $\triangle DBE$ forms an infinite number of similar triangles. Since corresponding parts of similar triangles are proportional, the ratio of the length of $\overline{DE}$ to the distance from B to $\overline{DE}$ will always be the same. This ratio will be the slope in the linear function.

Students who have taken high school geometry should understand and be able to discuss this geometric relationship. Students without this background should also understand it, albeit less formally.

2. Students need to be sure to click on existing endpoints of segments when drawing the second and third sides of the triangle. At the end of this step, they should have three segments and three points (labeled A, B, and C) in their sketches. (If students make and then delete mistakes, their labels may no longer match the descriptions here. In this case they should use the **Text** tool to change the labels to match the directions.)
3. Make sure $\overline{AB}$ is highlighted before you click. Drag the new point with the **Arrow** tool after this step to confirm that it moves only along the segment.
5. An alternate way to construct the point of intersection is to select the line and the segment and choose **Construct | Intersection.**
6. To hide the line, select it and choose **Display | Hide Line.** Use the **Segment** tool to construct $\overline{DE}$, or select D and E and choose **Construct | Segment.**

FUNCTIONAL RELATIONSHIPS

Q1 No matter what you drag, it's not possible to make one measurement increase while the other decreases. This suggests that there's a mathematical rule governing the behavior of the measurements.

Q2 Either choice is acceptable, and class discussion will be improved if different students make different choices here. Most students will probably choose B to $\overline{DE}$ as the independent variable because that measurement seems to change more directly as point D moves up and down $\overline{AB}$.

Q3 Answers will vary. Encourage students to put thought into their prediction. Ask them to explain their choices in writing or to compare and discuss them in groups. The graphs of students who chose different independent variables will be especially interesting throughout the activity.

Q4 The graph is a straight line—or, more specifically, a straight line segment—with one endpoint at the origin and the other somewhere in the first quadrant. The fact that the graph is linear tells you that as one quantity changes at a consistent rate, so does the other, though one may change faster than the other. The fact that it goes through (or at least up to) the origin means that when the distance between B and $\overline{DE}$ is zero, so is the length of $\overline{DE}$, and vice versa.

Q5 For choice a (the distance from B to $\overline{DE}$ is the independent variable), the domain is the set of distances from zero to the height of $\triangle ABC$, inclusive. The minimum distance, zero, occurs when D is at B. The maximum value corresponds to the height of $\triangle ABC$ and occurs when D is at A and $\overline{DE}$ coincides with $\overline{AC}$. For choice b, the above is true of the range.

For choice a, the range is the set of lengths from zero to the length of $\overline{AC}$, inclusive. The minimum value, zero, occurs when D is at B. The maximum value is equal to the length of $\overline{AC}$ and occurs when D is at A and $\overline{DE}$ coincides with $\overline{AC}$. For choice b, the above is true of the domain.

Q6 For choice a, the wider the base is relative to the height, and the shorter and wider the triangle is, the steeper the slope will be. (For choice b, the slope will be less steep.) This is true because the length of $\overline{DE}$ changes very quickly for very small changes in the distance between B and $\overline{DE}$. The smaller the base is relative to the height, and the taller and skinnier the triangle is, the less steep the slope will be for choice a. (It will be more steep for choice b.) This is true because as the distance from B to $\overline{DE}$ changes a lot, the length of $\overline{DE}$ changes much more slowly. Since the locus of choice b behaves like a reflection of choice a, the opposite of the preceding will be true for choice b.

Furthermore, any pair of similar triangles would produce graphs with the same slope. The only difference is that the larger triangle would have a larger domain and range.

EXPLORE MORE

13. Since the slope of the segment is the ratio of the two measurements, the function to plot is

$f(x) = (m\overline{DE}/\text{Distance } B \text{ to } \overline{DE}) \cdot x$ for choice a, and

$f(x) = (\text{Distance } B \text{ to } \overline{DE}/m\overline{DE}) \cdot x$ for choice b.

To make the plot coincide exactly with the trace of the plotted point, change the function's domain using **Edit | Properties | Plot.**

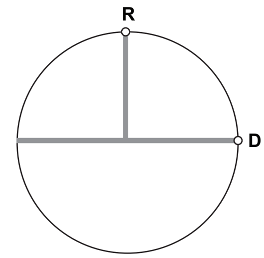
Q7 These plots will be similar to the original plot—segments from the origin—but with slopes less than or equal to that of the original plot. This is because the shortest distance from B to $\overline{DE}$ is along a perpendicular, and this is how the distance from B to $\overline{DE}$ is measured. Since the ratio corresponding to the slope of the plots has BD, BE, or the distance from B to $\overline{DE}$ in the denominator, the shortest distance corresponds to the greatest slope. The slopes would be equal for right triangles. (For example, if C were a right angle, the length of $\overline{BE}$ would equal the distance from B to $\overline{DE}$, and these two plots would coincide.)

Q8 Relationships between linear values (height, for example) and the area will be quadratic.

Functional Geometry

In many geometric figures, different aspects or quantities change in related ways. When two things change in a related way, their connection is called a *relation*. For example, the diameter and radius of a circle change together, so they form a relation.

Sometimes the second thing in a relation not only changes with the first, but actually depends on it. In this case we say the relation is also a *function*. In this activity you'll explore relations between geometric measurements and determine which of those relations are functions.



RADIUS AND DIAMETER

Q1 Does the diameter of a circle depend on the radius? Is the relation between the diameter and the radius a function? Explain your reasoning.

Q2 Does it also work the other way? Can you also think of the radius as a value that depends on the diameter? Is this relation a function? Explain.

1. Open **Functional Geometry.gsp.** Familiarize yourself with the figure by dragging each of the two points.

To measure these aspects of the circle, select each one at a time and choose **Measure | Length.**

2. Measure the circle's radius and diameter, and decide which measurement you want to control directly. This is the independent variable, and the other measurement is the dependent variable.

Q3 Which measurement will you use as the independent variable? Which point must you drag to change it?

Q4 As you drag your chosen point, observe how the two values change together, and try to imagine what a graph of the function would look like. Draw your prediction on your paper.

If the plotted point isn't visible, drag one of the tick numbers closer to the origin.

3. To test your guess, select the two measurements in order (independent variable first, then dependent variable) and choose **Graph | Plot As (x, y).** Drag your chosen point and observe the behavior of the plotted point.

4. Select the plotted point and choose **Display | Trace Plotted Point.** Observe the path of the plotted point as you drag your chosen point.

Q5 Compare your guess with the actual graph. Why is the graph the shape it is?

Q6 Think back to your decision to designate one of the measurements as the independent variable. Did it matter which one you chose? Why or why not? How are the graphs different? To answer this question, repeat steps 2–4, but this time make the opposite choice.

Q7 Based on what you've learned, is the relation a function? Why or why not?

INVESTIGATE

This sketch has eight more geometric figures on the remaining pages. On each page, explore how two different aspects of the figure relate to one another.

Measure two aspects of the figure that you're interested in exploring, and use these measurements to complete steps 2–4 and to answer Q3 and Q4.

Compare your prediction with Sketchpad's graph of the function. Why did the graph take the form that it did? Did it matter which measurement you designated as the independent variable? Why or why not?

Repeat the process, but this time make the opposite choice, and compare the two graphs.

Knowing what you've learned about the two aspects you've chosen in the figure, is either of the relations a function? Explain your answer.

SOME FUNCTION SUGGESTIONS

Page	**Suggested Measurements**
Circle	diameter, radius, circumference, area
Triangle	*AD*, *DE*, distance from *A* to *DE*
Xquad	*AD*, *DE*
Rect 1	*AD*, *AB*, perimeter, area
Rect 2	*AD*, *AB*, perimeter, area
Polygons	*AD*, area, perimeter for each polygon (Compare the graphs.)
Ladder	*AD*, *AB*
Hanger	*AD*, $\angle BDC$, area, perimeter
Tube View	tube length, tube diameter, distance, field of view

EXPLORE MORE

5. See if you can write an equation for one or more of the functions you graphed. Plot that function (choose **Graph | Plot New Function**) to see if it matches the locus graph (choose **Construct | Locus**). You might need to measure things in the sketch to use as parameters in your function.

Objective: Students explore relations defined by geometric measurements. They create graphs of the relations, explain how they decided which would be the independent and which the dependent variable, and determine whether the relation is a function.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should be familiar with graphing in the *xy* plane and with functions. Several other activities (including The Circumference Function and Functions in a Triangle) can be used to introduce these topics.

Sketchpad Level: Intermediate. Students measure objects, plot the resulting measurements, and create a graph.

Activity Time: 25–50 minutes. The time required depends on how much exploration students do. Try to allow an entire class period, and encourage students to explore and discuss their own functions and to plot equations to match their findings.

Setting: Paired/Individual Activity (use **Functional Geometry.gsp**)

Use this activity to encourage students to explore examples of relations and functions on their own. They will find many interesting functional relationships. There are even some quartic (4th-degree) polynomials lurking for those curious enough to find them.

INDEPENDENT AND DEPENDENT VARIABLES

This activity provides an excellent opportunity to explore and discuss the terms *independent variable* and *dependent variable*, so make sure that students plot each pair of quantities both ways, first with one quantity as the independent variable, and then with the other.

In the class discussion, point out that the terms have both a strict mathematical meaning and a common meaning. Mathematically, the terms describe the input and output values of a function, involving no causality beyond the association of each input value with a single output value. Mathematically, that's the end of the story; there's no deeper mathematical meaning in these terms.

However, the terms *independent* and *dependent* prompt students to think beyond the strict mathematical meaning and consider real-world causality. Can they change the thing being measured by the independent variable, and in doing so cause a change in the thing being measured by the dependent variable? Such a physical (or geometric) dependence is more complex, more ambiguous, and more interesting than the precise and limited mathematical connection between independent and dependent variables.

Encourage students to argue about which variable should be independent and which dependent for particular combinations of values. Arguments will relate mainly to the non-mathematical meanings of the words. Point this out, and ask students to keep in mind also the mathematical meanings. They should end up with a sense of the distinction between the mathematical and non-mathematical meanings of these terms. In the mathematical realm, they should realize that the question of whether a particular pair of quantities can be considered independent and dependent variables is precisely the question of whether the second is a function of the first: Does the second measurement ever have more than one value corresponding to any particular value of the first?

The circle model on page 1 is a good place to start this discussion. Ask students whether the radius really depends on the diameter, or the diameter on the radius. The relation between the radius and the diameter is intentionally ambiguous; you can drag either and the other changes. You will likely have students willing to expound on either side of the argument, as well as some who will claim that both are correct. Also discuss the function relating the circumference to the radius, and challenge students to describe physical scenarios in which the circumference (or even the area) can be varied independently, with the radius depending upon it.

Another topic for the discussion is linear and nonlinear relationships. When do you expect the graph to be a line, and when do you expect it to be a curve? You might expect a line when plotting lengths against other lengths—such as a side length and the perimeter of a figure—and a curve when plotting a length against an area, but there are plenty of exceptions for students to encounter and explore.

CONSTRUCTION TIPS

Students will want to control the appearance and scale of their graphs. Show them how to do each of these tasks:

- Drag the origin using the **Arrow** tool.
- Hide the grid by choosing **Graph | Hide Grid.**
- Change the scale by dragging the tick mark numbers.
- Allow independent scaling of each axis by choosing **Graph | Grid Form | Rectangular.**

RADIUS AND DIAMETER

Q1 The diameter depends on the radius: Given a radius, there's only one possible value for the diameter.

Q2 It works both ways: Given a diameter, there's only one possible value for the radius. Both the relation and its inverse are functions, because if you know either of the two values, you also know the other.

Q3 Students can use either measurement in either role.

Q4 The graphs that students predict will vary. The crucial point is that they make a prediction ahead of time.

Q5 The graph shows direct variation with a slope of 2 or 0.5, depending on the choice of independent variable. The equations can be written *radius* = *diameter*/2 or *diameter* = 2 • *radius.*

Q6 Either measurement can be the independent variable. Both graphs are linear, but they have different slopes.

Q7 Both relations are functions, because there's only a single output value possible for any input value.

INVESTIGATE

Circle: A relation of the area and any linear measure results in half of a parabola, opening up or to the right depending on which is used for x and which for y. Because the measures are positive, these are all functions.

Triangle: (AD, DE) is direct variation (a linear function through the origin). The constant of proportionality (the slope) depends on the triangle's shape. For the original shape, the constant is approximately 0.92.

(*Distance A* to DE, DE) is also direct variation, with a constant that depends on the shape of the triangle.

Xquad: (AD, DE) is a function; the graph is in the shape of a V because it involves an absolute value. (DE, AD) is not a function: There are two possible values for AD that correspond to any given distance DE.

Rect 1: (AD, AB) is a function. $ABCD$ is a constant-area rectangle, so the two sides are related by inverse variation, and the graph is in the shape of a hyperbola. Here's one form of the equation: $AB = (10/AD)$. This is one graph that's unchanged when students switch the dependent and independent variables.

(AD, *Perimeter ABCD*) is also a function, though (*Perimeter ABCD*, AD) is not. Here's the equation for the perimeter: *Perimeter ABCD* $= 2(10/AD) + 2AD$.

(AD, *Area ABCD*) is a constant function. *Area ABCD* $= 10$.

Rect 2: (AD, AB) is a linear function with a slope of -1. This is a constant-perimeter rectangle, so the sum of the lengths of the two sides stays constant: $AB = -AD + 10$.

(AD, *Perimeter ABCD*) is a constant function. The value of the constant depends on the length of a hidden segment in the construction.

(AD, *Area ABCD*) is a function, with a graph in the shape of a parabola opening down. (*Area ABCD*, AD) is not a function, because any value of the area (except when $ABCD$ is a square) corresponds to two possible positions of D.

Polygons: For each polygon, the area is a function of side length, and the graph is in the shape of a half-parabola in Quadrant I. The graph is of form *Area* $= a\,AD^2$, where the value of a depends on the number of sides. For a triangle, $a = 0.5$; for a square, $a = 1$; for a pentagon,

$$a = \frac{5\varphi^2}{4\sqrt{\varphi^2 + 1}} = \frac{\sqrt{25 + 10\sqrt{5}}}{4} \approx 1.72$$

where φ is the golden ratio.

The perimeter of each polygon varies directly with the side length: *Perimeter* $= n \cdot AD$, where n is the number of sides.

Ladder: The relation (AD, AB) is a function. The graph is a quarter circle centered at the origin: $AB = \sqrt{10^2 - AD^2}$

Hanger: Because the four sides of the hanger are fixed in length, there are a number of constant functions here. The area is a function of AD and can be expressed as the square root of a quartic, related to Hero's formula.

Tube: The *field of view* varies directly with *distance*, and also varies directly with *tube diameter*. It varies inversely with *tube length*.