Quadratic Intercepts

Consider a quadratic function expressed in this form:

$$y = ax^2 + bx + c$$

The graph of this function must have exactly one y-intercept, but the number of x-intercepts may be 0, 1, or 2. In this activity you'll assume a quadratic function with two x-intercepts. Your objective is to derive the function from the y-intercept and the two x-intercepts.

SUMS AND PRODUCTS OF ROOTS

The roots of a quadratic function, if they exist, are often labeled r_1 and r_2. They can be derived from the quadratic formula:

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

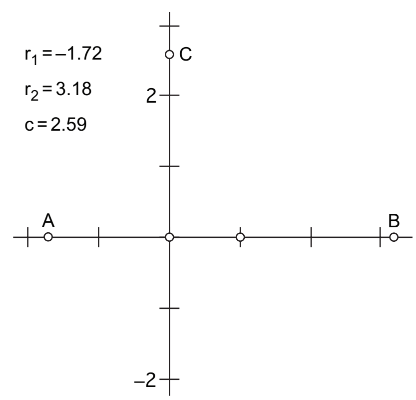
The formulas for the roots themselves can be cumbersome, but the formulas for their sum and product are simpler:

$$r_1 + r_2 = -\frac{b}{a} \qquad r_1 r_2 = \frac{c}{a}$$

Q1 Prove algebraically that the above formulas for the sum of the roots and the product of the roots are correct.

SKETCH AND INVESTIGATE

1. In a new sketch, choose **Graph | Define Coordinate System.** Construct points A and B on the x-axis and point C on the y-axis. These points will be the intercepts of the quadratic graph.



To put a subscript at the end of a label, type the subscript in square brackets, like this: r[1].

2. Select points A and B. Choose **Measure | Abscissa (x).** Change the measurement labels to r_1 and r_2. These are the roots of the function.
3. Select point C. Choose **Measure | Ordinate (y).** Change the measurement label to c (lowercase). This is c in the quadratic function.

Q2 Explain why the y-intercept corresponds to c in the quadratic function.

Q3 Start with the product-of-roots formula. Solve the equation for a. What is a in terms of r_1, r_2, and c?

4. Choose **Measure | Calculate.** Referring to your answer to Q3, use the measurements to calculate *a.*

Q4 Now move to the sum-of-roots formula. Solve the equation for *b*. What is *b* in terms of r_1, r_2, and *a*?

5. Referring to your answer to Q4, use the measurements to calculate *b.*

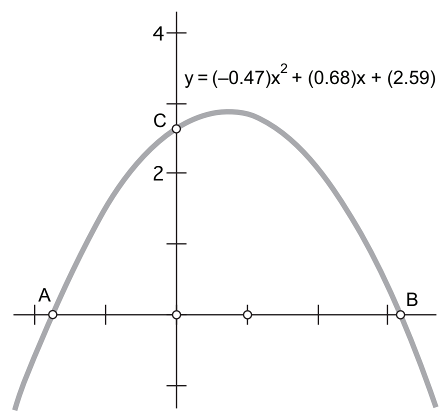
6. At this point you should have *a, b,* and *c* on the screen. Choose **Graph | Plot New Function.** Enter $ax^2 + bx + c$.

Q5 Drag the points and see if the graph fits them for all cases. Are there any positions for points *A, B,* and *C* for which this graph construction does not work?

Q6 Is it possible to graph any and all quadratic functions by moving points *A, B,* and *C* to certain positions?

PRESENT

Now that you have fit the graph to the points, complete the presentation by writing the function using a custom template.



7. Use the text tool to create the following caption:

$$=y = (\{1\})x^2 + (\{2\})x + (\{3\})$$

8. Select, in order, the caption, followed by the measurements *a, b,* and *c*. Choose **Edit | Merge Text.**

Objective: Students learn the sum-of-roots and product-of-roots formulas for quadratic functions, and use them to derive a quadratic function from three intercepts.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should first understand the concepts of roots, interpreted graphically and analytically. They should be familiar with the quadratic formula.

Sketchpad Level: Intermediate. Most of the work involves creating calculations in Sketchpad.

Activity Time: 30 minutes

Setting: Paired/Individual Activity (no sketch needed) or Whole-Class Presentation (use **Quadratic Intercepts Present.gsp**)

SUMS AND PRODUCTS OF ROOTS

Q1

$$r_1 + r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} + \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2b}{2a}$$

$$= -\frac{b}{a}$$

$$r_1 r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{(2a)^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}$$

SKETCH AND INVESTIGATE

Q2 The y-intercept point is on the y-axis, and therefore has x-coordinate 0. Substitute 0 for x in the general form.

$$y = a(0)^2 + b(0) + c$$
$$= c$$

The y-intercept is at $(0, c)$.

Q3 $a = \frac{c}{r_1 r_2}$

Q4 $b = -a(r_1 + r_2)$

Q5 If any of the intercept points falls on the origin, then at least one root is 0, and the formula from Q3 is undefined. This does not mean that no quadratic graph includes the origin; it means that more information is needed in order to derive the function and graph it.

Q6 Quadratic functions having no roots cannot be graphed using this method, because r_1 and r_2 would both be undefined.

This method works for any quadratic function having two roots, with the exception of the case described in Q5. The graph cannot include the origin.

In the case of a single root, points A and B are coincident, making $r_1 = r_2$. That does not conflict with any of the calculations, so it is still possible to derive the function and plot it. Again, the graph must not include the origin.

This activity uses the sum-of-roots and product-of-roots formulas to derive a quadratic function from the axis intercepts of its graph.

1. Write a general form for a quadratic function, and also the formulas for the quadratic roots, the sum of the roots, and the product of the roots:

$$y = ax^2 + bx + c \qquad r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad r_1 + r_2 = -\frac{b}{a}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad r_1 r_2 = \frac{c}{a}$$

2. Open **Quadratic Intercepts Present.gsp.** Drag points *A*, *B*, and *C* in turn. These represent two *x*-intercepts and the *y*-intercept of the quadratic graph. The corresponding values of the intercepts already appear as measurements on the screen.

Q1 Why do the roots, r_1 and r_2, match the *x*-intercepts? (Because at those points the function is equal to zero.)

Q2 In order to derive the function, you need to find the coefficients *a*, *b*, and *c*. How can you use the intercept values and the root sum and difference formulas to help?

3. Press the *Show* buttons in turn to go through the steps of deriving the coefficients. Pause before each step to allow students to offer answers. If you do them in the right order (*c*, *a*, *b*), the calculations are fairly easy. These are the same calculations used for the function plot.

$$c = y_c \qquad a = \frac{c}{r_1 r_2} \qquad b = -a(r_1 + r_2)$$

Q3 What if the graph goes through the origin? (Press *Set r2 = 0 and c = 0.*) The graph disappears. Does that mean that a quadratic function cannot include the origin? Are the sum and product formulas valid in this case?

4. Take some time for discussion. Students may see that if $r_2 = 0$, then the formula used for coefficient *a* involves division by zero.

5. Go to page 2. This is a different construction of a quadratic function including the origin. This time the parabola vertex is showing. Drag the vertex up and down to show a family of functions. In fact, the roots and the function coefficients are defined for all of these cases. The problem was not that there were no solutions, but that there were too many solutions. Two points are not enough to define a quadratic function.