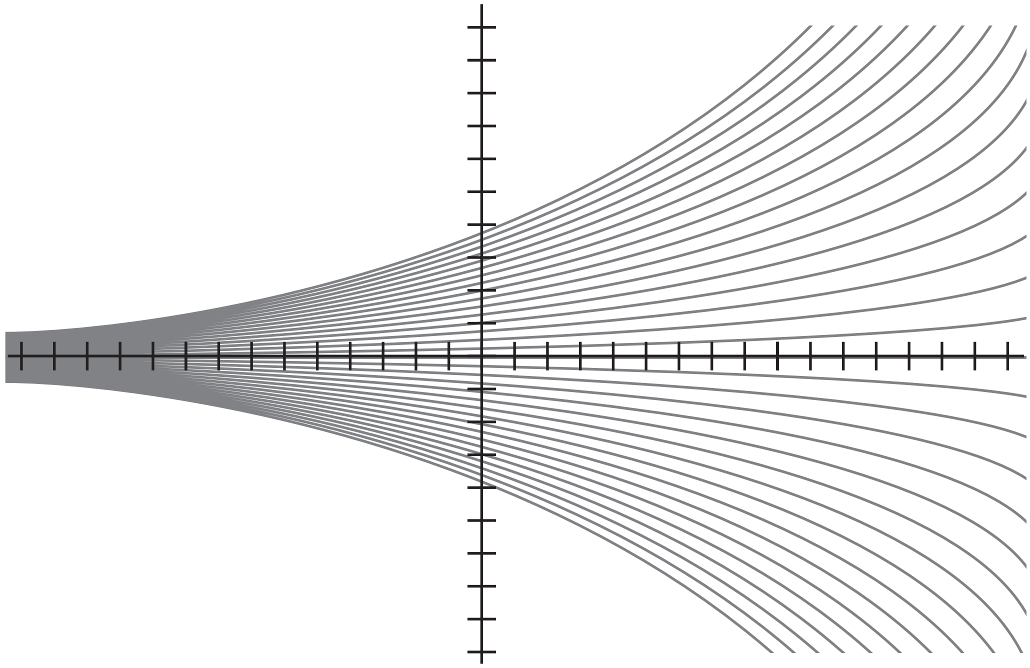
6

Other Functions



Absolute Value Functions

The absolute value of a number is how big the number is, regardless of whether it's positive or negative. Some examples should make this clearer.

The absolute value of -5 is 5, *or* $|-5| = 5$.

The absolute value of 5 is 5, *or* $|5| = 5$.

The absolute value of 0 is 0, *or* $|0| = 0$.

As you can see, the absolute value of a number is always a positive number or zero.

But what happens when you graph an equation involving the absolute value function, such as $y = |2x - 4|$? In this activity you'll find out.

SKETCH AND INVESTIGATE

Choose **Graph | Plot New Function** to open the New Function calculator. Then type x and click OK.

1. In a new sketch, plot the equation $y = x$.
2. Think about what the graph of $y = |x|$ might look like. Plotting points by hand or discussing the question with classmates might help. Make a rough sketch of your guess on scratch paper.

Choose **Plot New Function** again. Then choose **abs** from the Functions pop-up menu, enter x, and click OK.

3. Plot the equation $y = |x|$. How does it compare with your prediction?
4. Repeat steps 1–3 with the equations $y = 2x - 4$ and $y = |2x - 4|$. Make sure to draw a prediction of what you think the second equation will look like before plotting it in Sketchpad.

Q1 Describe how the graphs of $y = 2x - 4$ and $y = |2x - 4|$ compare. Discuss their shapes, their ranges, and other features you notice.

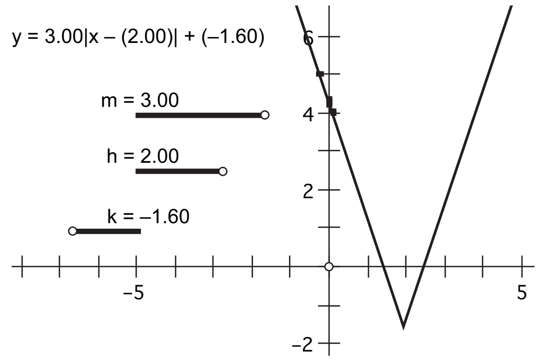
Q2 Predict what the graph of $y = |2x| - 4$ will look like. After making your prediction, plot it. Then describe the differences between this graph and the other two.

A FAMILY OF ABSOLUTE VALUE GRAPHS

As you explore this family, keep in mind a related family: lines in point-slope form $y = m(x - h) + k$.

Now that you have an idea of what absolute values can do to the graphs of particular functions, you'll explore a *family* of graphs: $y = m|x - h| + k$.

5. Open **VGraph.gsp.** You'll see the graph of an equation in the form $y = m\,|\,x - h\,| + k$, and sliders for the parameters *m*, *h*, and *k*.



Q3 Adjust slider *m* and observe the effect this has on the V-graph. Try different values for the slope—large, small, positive, negative, and zero. Summarize the role *m* plays in the equation $y = m\,|\,x - h\,| + k$.

Q4 Changing *m* moves the entire V-graph except for one point. This point is called the *vertex*. Adjust the sliders for *h* and *k*. How does the location of the vertex relate to the values of *h* and *k*?

Q5 Write an equation in the form $y = m\,|\,x - h\,| + k$ for each of the V-graphs described below. Check each answer by adjusting the *m*, *h*, and *k* sliders.

a. Vertex at $(-1, 2)$; contains the point $(0, 4)$

b. Vertex at $(2, 3)$; contains the point $(4, 0)$

c. Vertex at $(-3, -1)$; same shape as $y = 3\,|\,x - 2\,| + 5$

d. *x*-intercepts at $(4, 0)$ and $(-4, 0)$; contains the point $(1, -2)$

e. *x*-intercepts at $(6, 0)$ and $(-2, 0)$; contains the point $(5, 1)$

f. Same vertex as $y = 3\,|\,x - 2\,| + 5$; contains the point $(0, 0)$

Q6 Not all graphs in this family have *x*-intercepts. How can you tell whether a function has *x*-intercepts just by looking at the function parameters?

EXPLORE MORE

To enter sin(*x*), choose **sin** from the Calculator's Functions pop-up menu. For this function's graph to appear properly, Sketchpad's angle units must be set to radians. Use **Edit** | **Preferences** | **Units** to check this setting.

Q7 Plot the following two pairs of equations:

a. $y = x^2 - 1$ and $y = |\,x^2 - 1\,|$

b. $y = \sin x$ and $y = |\sin x|$ (Don't worry if you're unfamiliar with the sin function.)

Write a short paragraph summarizing what happens when you plot a function and its absolute value.

Objective: Students graph the absolute value function and various transformations. In the process they review the point-slope form of linear functions and prepare for the vertex form of quadratic functions.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should know the definition of the absolute value function, but they need no experience with the graph.

Sketchpad Level: Easy

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **VGraph.gsp**) or Whole-Class Presentation (use **VGraph Present.gsp**)

Related Activities: Parabolas in Vertex Form

In addition to familiarizing students with various transformations of the absolute value graph, there are two other reasons for doing this activity:

First, it reinforces students' understanding of the point-slope form of lines (with the role of point (h, k) becoming even more apparent).

Second, it prepares students for the vertex form of parabolas.

It's important for students to actually predict what the absolute value graphs will look like before they graph them. Just by plotting a few points—such as $(-2, 2)$, $(0, 0)$, and $(2, 2)$—they can gain insight into why these graphs appear as they do.

SKETCH AND INVESTIGATE

Q1 The graphs are identical to the right of $x = 2$. To the left of $x = 2$, the graphs are reflections on the x-axis. Another way of describing this is that the part of $y = 2x - 4$ that was below the x-axis has been flipped above the x-axis. The graph of $y = 2x - 4$ is a line whereas the graph of $y = |2x - 4|$ is shaped like a V. The range of $y = 2x - 4$ is all real numbers whereas the range of $y = |2x - 4|$ is all real numbers greater than or equal to zero.

Q2 This graph has the same parent function ($y = 2x - 4$), but a different vertex: the point $(0, -4)$. The purpose of this question is to spur students to think about how the numbers in the equation relate to the position of the vertex.

A FAMILY OF ABSOLUTE VALUE GRAPHS

Q3 The sign of m determines whether the V opens upward or downward or is flat. The graph of an equation with a positive m opens upward; with a negative m, it opens downward; and with $m = 0$, it is straight and horizontal. The greater the magnitude of m, the smaller the angle at the vertex. The slope of the right side of the V always equals m while the slope of the left side always equals $-m$.

Q4 The coordinates of the vertex are (h, k).

Q5
a. $y = 2|x + 1| + 2$ b. $y = -1.5|x - 2| + 3$
c. $y = 3|x + 3| - 1$ d. $y = (2/3)|x| - 8/3$
e. $y = -1|x - 2| + 4$ f. $y = -2.5|x - 2| + 5$

Q6 The graph does not have x-intercepts if parameters m and k have the same sign. For example, if k is positive, then the vertex is above the x-axis. If m is also positive, then the graph opens upward, so it can never reach down to the x-axis.

If m and k have opposite signs, then there are two x-intercepts. If $k = 0$, then the vertex itself is an x-intercept.

EXPLORE MORE

Q7 Consider the graphs of two equations: $y = f(x)$ and $y = |f(x)|$. Where the first graph is below the x-axis, it will be reflected across (above) the x-axis in the second graph. Where the first graph is above the x-axis, the second graph will be identical.

1. Open **VGraph Present.gsp.** Press *Show* $y = x$. After a short pause, press *Show* $y = |x|$.

Q1 It looks like the graphs coincide on the right side. Geometrically, what is their relationship on the left? (They are reflections on the x-axis.)

There is also a reflection on the y-axis. If anyone mentions that, acknowledge it and verify that reflection too.

2. To verify this, construct a point on the left side of either graph. Double-click the x-axis to mark it as a mirror. Select the point and choose **Transform | Reflect.** Drag the first point to show that the image follows the other graph.

3. On page 2, press *Show* $y = mx + b$. Drag the two sliders to show how they control the graph.

Q2 Ask, "What will the graph of $y = |mx + b|$ look like?" Discuss predictions and show the graph.

Q3 Give m a positive value for this question. What is the slope on the right part of the absolute value graph? (m) What is the slope on the left side? ($-m$)

4. The slope on the right side should be clear because of the coinciding graphs. To check the slope on the left side, construct a line segment with both endpoints on the left side of the absolute value graph. Select the segment and choose **Measure | Slope.**

5. On page 3 is the graph of $y = m|x|$, with $m = 1$ to begin.

There is a button that will show m as a fraction, reinforcing the slope concept.

Q4 Before moving the slider, ask for predictions. Then vary m and ask for observations as you make m bigger, smaller, negative, and zero. (As with multiplying any other function by a constant, m defines the ratio for a vertical stretch. In this case an m with larger magnitude makes the vertex angle smaller. If m is positive, it opens upward; if negative, downward. When $m = 0$, the graph coincides with the x-axis.)

6. Page 4 contains three sliders and the graph of $y = m|x - h| + k$.

Q5 What are the effects of changing the parameters? Parameters h and k define horizontal and vertical translation. Parameter m has the same effect as before.

Q6 Identify the vertex. What are the coordinates of this point? (h, k) To emphasize this, press *Reset,* and then drag the h and k sliders one at a time. Give students the coordinates of the vertex and one other point, and challenge them to derive corresponding settings for m, h, and k.

7. Page 5 has the graph of a general function along with its absolute value. Edit the definition of $f(x)$ and have students predict the shape of its absolute value. Challenge them with functions they have never seen. Given the root function graph, they should still be able to predict the shape of the absolute value.

Exponential Functions

There is a connection between population growth, radioactive decay, musical scales, and compound interest. They seem to have little in common, but you can model any of them using an exponential function.

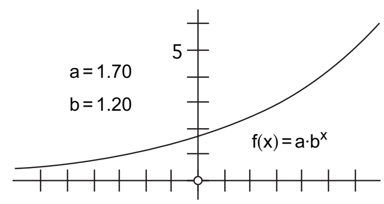
An exponential function has the general form $f(x) = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$.

PROPERTIES OF THE GRAPH

Before using an exponential function to model a real-world problem, take some time to familiarize yourself with the graph.

To create a parameter, choose **Graph | New Parameter.**

1. In a new sketch, create parameters a and b.
2. Graph the function $f(x) = a \cdot b^x$ by choosing **Graph | Plot New Function.** Click the parameters in the sketch to enter them into the function.



To change the value of a parameter, either double-click it or select it and press the + or − key. (To change the size of the steps for the + or − keys, select the parameter and choose **Edit | Properties | Parameter.** Change the Keyboard Adjustments value to 0.1 unit.)

3. The graph is plotted on the screen. Change the values of the parameters, and observe the resulting changes in the graph. Try several values for each parameter.

Q1 What are the x- and y-intercepts of the graph? Explain how the intercepts are related to parameters a and b.

Q2 The value of $f(x)$ tends to get close to zero either on the left side or on the right. What parameter values determine which side it is?

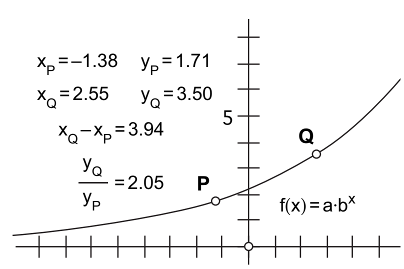
Q3 In the general form of the exponential function, there are three constraints ($a \neq 0$, $b > 0$, and $b \neq 1$). Explain the reason for each of these constraints.

Next you'll investigate how the function behaves by comparing the coordinates of two points on the graph.

4. Change the parameters so that $a = 2.00$ and $b = 1.30$.
5. Construct two points on the function graph. Label them P and Q.
6. Select both points and choose **Measure | Abscissa (x).** Select the points again and choose **Measure | Ordinate (y).**
7. Calculate the values $x_Q - x_P$ and y_Q/y_P.

You can change the scale of the axes to give you more precise control over the positions of the points.

Q4 Drag point Q one unit to the right of P, so that the difference $x_Q - x_P$ is as close to 1.00 as you can make it. What is the value of the ratio y_Q/y_P? Drag point P to a different



position on the graph, and again drag Q so it's one unit to the right. What is the value of the ratio y_Q/y_P? Why do you get this result?

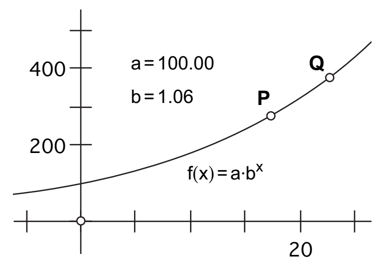
Q5 Now use a difference other than 1. Drag the points so that $x_Q - x_P$ is approximately 2.5. What is the ratio? Drag them to another position, but with the x difference still equal to 2.5. What is the ratio now? Explain.

DOUBLING PERIOD AND HALF-LIFE

Exponential functions can be used to solve a number of real-life problems. First change the function so that it shows the value of $100 invested at an effective annual yield of 6%.

Effective annual yield is not the same thing as interest rate. That's another topic.

8. Edit parameters a and b so that the function has the definition $f(x) = 100(1.06)^x$. This shows the value of $100 invested at an effective annual yield of 6%. (The x variable is in years, and y is in dollars.)



Setting the grid to rectangular allows you to adjust each axis independently of the other.

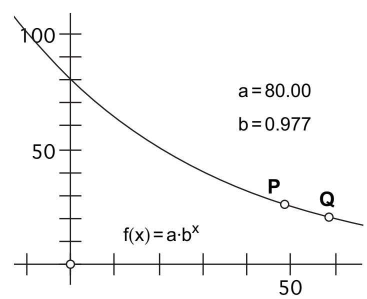
9. At first you can't see the graph because the y-axis doesn't go up to 100. To adjust the axes, choose **Graph | Grid Form | Rectangular Grid.** Then drag tick mark numbers on each axis so that you can see the results for the first 25 years.

To see more decimal places in a parameter, select the parameter and choose **Edit | Properties | Value.** Change the Precision setting.

Q6 How long will it take to double your money? Drag the points so that the ratio is 2.00. What is the difference in their x-coordinates? This number is called the *doubling period.*

An exponential function can also be used to model the decay of radioactive cesium.

10. To model the decay of 80 g of cesium, change the function definition to $f(x) = 80(0.977)^x$. Adjust the axes appropriately. The value of x is still in years, but the value of y is in grams.



Q7 If you start with 80 g, you will have less cesium every year. How long would it take to lose half of it? Explain how you found the answer. This number is called the *half-life* of cesium.

Q8 Although cesium decays, as opposed to growing, you can still calculate its doubling period. Drag the two points until you find a position where the ratio is 2.00. What is the difference in the x-coordinates? Explain how this verifies your answer to Q7.

Objective: Students graph exponential functions and examine their properties. They use exponential functions to model compound interest and radioactive decay.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students must first understand how to work with expressions with exponents.

Sketchpad Level: Intermediate. Students create their own sketches using the graphing tools.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Exponential Present.gsp**)

PROPERTIES OF THE GRAPH

Q1 Both a and b are nonzero. Therefore, ab^x is also nonzero for any real x, so y cannot be zero, and there is no x-intercept.

For the y-intercept, substitute zero for x in the equation $y = ab^x$.

$$y = ab^0 = a$$

The y-intercept is a.

Q2 For $b > 1$, $f(x)$ tends to zero on the left. For $0 < b < 1$, $f(x)$ tends to zero on the right. The value of a has no influence on this property.

Q3 If a were equal to zero, the function would be the constant function $f(x) = 0$.

If b were equal to zero, the function would be zero for all positive x, and it would be undefined for all other values of x.

If b were less than zero, b^x would not be continuously defined over all real exponents x.

If b were equal to one, this would be another constant function, $f(x) = a$.

Q4 Limited resolution may prevent students from making the difference exactly 1.00. It's sufficient for them to make it as close to that value as they can. The ratio $y_Q/y_P = 1.30$. This is the same as parameter b.

$$\frac{y_Q}{y_P} = \frac{f(x_Q)}{f(x_P)} = \frac{ab^{x_Q}}{ab^{x_P}} = b^{x_Q - x_P} = b^1$$

Here it does not matter where on the graph points P and Q are, so long as $x_Q - x_P = 1$.

Q5 If $x_Q - x_P = 2.5$, then $y_Q/y_P = 1.30^{2.5} \approx 1.9$. This follows from the same reasoning as in the previous answer.

$$\frac{y_Q}{y_P} = \frac{f(x_Q)}{f(x_P)} = \frac{ab^{x_Q}}{ab^{x_P}} = b^{x_Q - x_P} = b^{2.5}$$

DOUBLING PERIOD AND HALF-LIFE

8. There may be some confusion regarding the term *effective annual yield*. The actual rate is about 5.8%, but since it is compounded continuously, at the end of each year, the investment will be worth 6% more than it was at the beginning of the year. Even if students do not yet grasp the concept, they can continue with the given function definition.

Q6 The doubling period is about 11.90 years.

Q7 To find the half-life, students should arrange the points so that $y_Q/y_P = 0.50$. When that happens, $x_Q - x_P \approx 30$, so cesium has a half-life of about 30 years. In this case changes in the scales of the axes can cause quite a lot of variation in the answers.

Q8 To find a doubling period, with the ratio $y_Q/y_P = 2.00$, point Q would have to be to the left of P. In that case, $x_Q - x_P \approx -30$.

This answer fits with the half-life answer. It stands to reason that if it takes 30 years for half of the cesium to decay, then 30 years ago, there was two times as much. This explains the negative doubling period.

1. Begin by showing the general form of this exponential function:

$$f(x) = ab^x, \quad \text{where } a \neq 0,\ b > 0, \text{ and } b \neq 1$$

Press *Show Slider Controls* to reveal buttons that you can use to set the parameters to various precise values.

2. Open **Exponential Present.gsp.** This is a graph of the function with sliders controlling the values of *a* and *b*. Drag each slider in turn so that students can see the effects on the graph.

Q1 Ask students for the *x*- and *y*-intercepts. (There is no *x*-intercept, and the *y*-intercept is equal to *a*.) Challenge them to verify these facts by alternately setting *y* and *x* to zero in the equation $y = ab^x$.

Q2 Sometimes the function approaches zero on the left side, and sometimes on the right. What determines which side it is? (It's on the left when $b > 1$ and on the right when $0 < b < 1$.)

Q3 In the general form of the function, there are three restrictions on the parameters *a* and *b*. Why? Show students what happens when $a = 0$, $b < 0$, $b = 0$, or $b = 1$.

3. On page 2 there are two points on the graph, *P* and *Q*. You can drag *P* freely, but point *Q* is a fixed distance to the right of point *P*. That distance is determined by the slider labeled Δ. At the bottom of the screen are measurements showing the difference of the *x*-coordinates and the ratio of the *y*-coordinates.

4. Drag point *P* to show that the ratio of the *y*-coordinates remains constant when the difference in the *x*-coordinates (Δ) is constant.

On pages 3 and 4, the *a* and *b* sliders have been replaced with parameters in order to make it easier to enter precise values.

Q4 What will the ratio be when $\Delta = 1.00$? (It should equal *b*.) Challenge students to predict this before showing it. Then have them prove that $\frac{f(x_P + 1)}{f(x_P)} = b$.

5. Page 3 has a graph showing the growth of an investment of $100 with an effective annual yield of 6%.

Pages 3 and 4 have rectangular grids. If students have not used that feature yet, this would be a good opportunity to show them the advantages of using different scales on the axes.

Q5 With this investment, how long would it take to double your money? If the money is doubled between *P* and *Q*, then the ratio of their *y*-coordinates will be 2. Drag the Δ slider until the ratio is 2.00. (The doubling period is the difference in the *x*-coordinates, about 11.9 years.)

6. The graph on page 4 shows the radioactive decay of 80 g of cesium. The *x*-scale is in years.

Q6 What is the half-life of cesium? This question is similar to the previous one. Give students time to figure out that they need to adjust the difference in the *x*-coordinates so that the ratio is 0.50. (The result is a half-life of about 30 years.)

Logarithmic Functions

Many occurrences in our natural world can be modeled using logarithmic functions, including the strength of earthquakes, the intensity of sound, or the concentration of hydronium ions in a solution. In this activity you'll explore the relationship between exponential and logarithmic functions, and determine how to write the formula for a logarithmic function that's the inverse of a particular exponential function.

GRAPH INVERSE EXPONENTIAL FUNCTIONS

Logarithms are related to exponents, so start by graphing an exponential function and finding the inverse graph.

1. Open **Logarithmic Functions.gsp.** Press the *Show Exponential Function* button to see the exponential function $y = 2^x$ along with its graph.

With the points selected, choose **Measure | Coordinates** to find their ordered pairs. With the coordinates selected, choose **Graph | Tabulate** to place the coordinates in a table.

2. Press the *Show Points* button to show seven points on the curve. Measure their coordinates, and put the resulting ordered pairs into a table.

Q1 Notice that some of the x-values are negative. Does this mean that the resulting values of the function are negative? Explain why this is true or not true.

Next, interchange the x- and y-values by reflecting the points over the line $y = x$.

3. Press the *Show y=x Line* button. With the line selected, choose **Transform | Mark Mirror.** The line flashes briefly to indicate that it is marked as the mirror.

To show the labels of the selected points, choose **Display | Show Labels.**

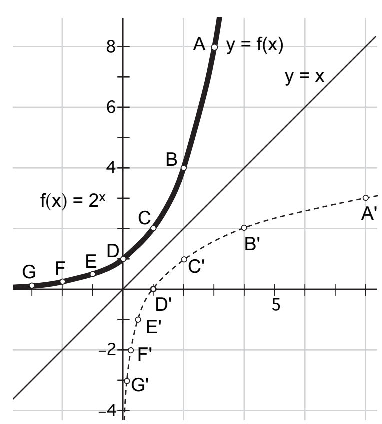
4. Press *Show Points* again to select the seven points in order, and choose **Transform | Reflect.** The seven points are reflected. Show their labels.
5. Measure the coordinates of the reflected points and tabulate the results. Align the two tables in order to see the original and reflected points next to each other.

Q2 What do you notice about the coordinates of each pair of points?

Q3 Will any of the x-coordinates of the reflected points be negative? Explain.

Q4 Why is the line $y = x$ called the *axis of symmetry* for a function and its inverse?

Next reflect the entire graph over the line $y = x$.



Use the **Point** tool to construct the new point.

6. Construct a point on the original graph, and reflect it over the line $y = x$. Drag your new point and observe the behavior of the reflected image.

The reflected graph is the graph of the inverse of the original function.

7. To create the entire reflected graph, select the point on the graph and its reflected image, and choose **Construct | Locus.** Change the color of the locus, and make it dashed.

You'll use page 2 to graph $y = 10^x$, reflect it to show its inverse, and compare the inverse to the graph of $y = \log x$.

8. On page 2 construct the graph of $y = 10^x$ by choosing **Graph | Plot New Function** and entering 10^x into the Calculator.
9. Construct a point on the graph and reflect it across the graph of $y = x$.

To turn tracing on or off, select the point and choose **Display | Trace Point.**

10. Turn on tracing for the reflected point, and drag the point on the graph to observe the shape of the inverse function.
11. Construct the graph of $y = \log x$.

Q5 What do you observe about the graph of the log function and the reflected image of the exponential graph? What conclusion can you draw?

On page 3 you'll graph the exponential function $f(x) = k^x$ and use different values for k to find a general formula for the logarithmic function that's the inverse of $f(x)$.

Choose **Graph | New Parameter** to create the new parameter, and set its label to *k* and its value to 2 in the dialog box that appears.

12. Create a parameter k, set its value to 2, and use it to construct the graph of $y = k^x$. To enter k into the function definition, click its value in the sketch.
13. Construct a point on the graph, reflect it across $y = x$, and construct the locus.

This locus is the graph of the inverse function. Express this inverse as a logarithmic function by stretching or shrinking the parent logarithmic function $y = \log x$.

14. Using the values of a and b, plot the logarithmic function $y = a \log(x/b)$. Adjust the sliders so that your newly plotted function matches the inverse of $y = k^x$.

Q6 What values of a and b made the graphs match?

To add a row to a table, making the current values permanent, double-click the table.

15. Record the values of k, a, and b in a table. Then change the value of k to 5, match the graphs again, and add a new row of data to the table. Continue adding new data to the table for the following values of k: 10, 100, and 1000.

Q7 What pattern can you find to relate the value of k to the values of a and b?

Q8 Use this pattern to predict the values of a and b needed when $k = 10{,}000$. Test your prediction by gathering another row of data for your table. Then predict the values needed when $k = 0.1$, and test your prediction.

Q9 Use your results to write a formula for the inverse of $f(x) = k^x$.

EXPLORE MORE

Q10 Use algebraic manipulation to explain why your formula from Q9 must be true.

Q11 A general exponential function can be written as $f(x) = a \cdot 10^{(x-h)/b} + k$. Write the corresponding inverse function in terms of a, b, h, and k.

Objective: Students explore the graphical relationship between exponential and logarithmic functions and determine how to write the formula for a logarithmic function that's the inverse of a particular exponential function.

Student Audience: Algebra 2

Prerequisites: Students should be familiar with exponential functions and with the properties of inverse functions.

Sketchpad Level: Intermediate/Challenging. This activity involves a fair amount of graphing and construction. Students should be able to repeat later in the activity a set of steps they used earlier.

Activity Time: 40–50 minutes. You can do steps 9–12 as a presentation to save time.

Setting: Paired/Individual Activity (use **Logarithmic Functions.gsp**) or Whole-Class Presentation (use **Logarithmic Functions Present.gsp**)

Related Activities: Exponential Functions

GRAPH INVERSE EXPONENTIAL FUNCTIONS

On page 1 students graph $y = 2^x$ to begin with an exponential function that has points whose coordinates they can verify in their heads. On page 2 they use a base of 10 to make it easy to see that 10^x and $\log x$ are inverse functions. On page 3 they use function transformations to find the inverse of a more general exponential function, and end up with a conjecture as to how to write the formula for such a function. This portion of the activity is useful either before introducing the change-of-base property of logarithms (to motivate that method) or afterward (to review the method).

2. When students press *Show Points,* the seven points are shown and selected. Because the points are selected in order, it's easy for students to measure and then tabulate the coordinates. Be sure they measure the coordinates using **Measure | Coordinates** rather than measuring the *x*- and *y*-values separately. If they deselect all objects at any point in the process, they can press *Show Points* to select objects again in order.

Q1 A negative value of x in the exponential function means that the exponent is negative. A positive base raised to a negative exponent results in a positive number.

Q2 Within each pair of points (A and A', B and B', and so forth), the *x*- and *y*-values are interchanged.

Q3 None of the *x*-coordinates of the reflected points can be negative, because none of the *y*-coordinates of the original function were negative.

Q4 The line $y = x$ is called the *axis of symmetry* because it lies precisely between the two graphs, reflecting each onto the other.

Q5 The graph of the log function coincides with the reflected image of the exponential graph, so the log function must be the inverse of the original exponential function.

13. This instruction is brief and does not give precise details concerning commands. If students have trouble with it, have them review what they did in steps 7–8.

Q6 The two graphs match when a is approximately 3.32 and $b = 1$. This value of $a = 3.32$ is approximately $1/\log 2$, but students don't have enough information yet to reach this conclusion.

Q7 When k is 10^1, $a = 1$; when k is 10^2, $a = 1/2$; when k is 10^3, $a = 1/3$. It appears that a must be $1/\log k$.

Q8 Because $10{,}000 = 10^4$, the value of a must be 1/4. Because $0.1 = 10^{-1}$, the value of a must be $1/-1 = -1$. Students should verify these predictions by testing them in the sketch.

Q9 The inverse of $f(x) = k^x$ is $f^{-1}(x) = \log x/\log k$.

EXPLORE MORE

Q10 Define $d = \log k$. Rewrite k as 10^d. Then:

$$y = k^x = (10^d)^x = 10^{dx}$$

$$\text{so } \log y = dx$$

$$\text{and } x = \frac{\log y}{d} = \frac{\log y}{\log k}$$

Q11 The inverse function is $f^{-1}(x) = b\log((x - k)/a) + h$.

In this presentation students will find the log function that's the inverse of $f(x) = k^x$.

Remind students that the ordered pairs of an inverse function are in the opposite order.

1. Open **Logarithmic Functions Present.gsp.** Press *Show Exponential Function.* The graph of $f(x) = 2^x$ appears. Press *Show Points* and then *Show Coordinates* to show seven points on the graph, along with their coordinates.

Q1 On the inverse function, what are the coordinates of the point corresponding to A on the original function? (8.00, 3.00) Ask different students to give coordinates for points on the inverse corresponding to each of the other points.

Q2 What geometric transformation do you know that switches the x- and y-values? (reflection across the line $y = x$)

2. Use the next three buttons to reflect the points across $y = x$ and show the coordinates. The two tables confirm the answers students gave for Q1.
3. To see the entire graph of the inverse function, press *Show Reflected Graph.*

Now compare an exponential function with a logarithmic function, and demonstrate that they are inverses by verifying that each is a reflection of the other across $y = x$.

4. On page 2, use the top two buttons to show the exponential function $f(x) = 10^x$ and the logarithmic function $g(x) = \log x$.
5. Use the next two buttons to show $y = x$, a point on $f(x)$, and its reflection.
6. Use the Animate button to animate the point, trace the reflection, and verify that the reflection (and therefore the inverse) of $f(x) = 10^x$ really is $g(x) = \log x$.

Now graph an exponential function with an adjustable base, and graph a stretchable logarithmic function. Match the logarithmic function to the inverse of the exponential function, and find a formula for the inverse of the exponential function.

7. On page 3, press the first three buttons to show the function $f(x) = k^x$ and its inverse. Use the green buttons to switch k from 2 to 5 and back to 2 again.
8. Show the adjustable logarithmic function, and drag sliders a and b to make the red logarithmic function match the blue inverse function.
9. Show the table of values for k, a, and b. Double-click the table to make the current values permanent.
10. Set k to the values 5, 10, 100, and 1000. For each value of k, drag sliders a and b to match the inverse, and record the values permanently in the table.

Q3 Ask, "How can you determine the values of a and b from k?" (The value of b is always 1, and the value of a is $1/\log k$.)

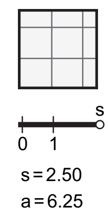
11. Test this conclusion by using the remaining green buttons to adjust k, predicting the required values of a and b, and testing the predictions by dragging a and b.

Square Root Functions

An important formula in geometry is the formula for the area of a square. In this activity you'll start with this formula to gain an understanding of the square root function and to explore transformations of this function.

INVESTIGATE

You'll use a model for the area of a square to begin your investigation.



1. Open **Square Root Fns.gsp.** This page shows, in two different ways, the relationship between the side and the area of a square.
2. Use the slider below the top square to drag *s* left and right.

Q1 Drag *s* so it's exactly 2.00. What is *a*? Make *s* exactly 5.00. What is *a* now? What happens to *a* when you move *s* to the negative side? Without using specific numbers, describe the relationship between *s* and *a*.

Q2 In this model, does *s* depend on *a*, or does *a* depend on *s*? Which of these two is the independent variable? How can you tell?

Q3 Write a formula for *a* in terms of *s*.

3. To plot a data point, select the two measurements in order (independent variable first, then dependent variable). Choose **Graph | Plot As (x, y),** and then choose **Display | Trace Plotted Point.** Label the point *P*.
4. To see the entire graph, vary the variable by dragging *s* left and right.

In the bottom square the roles of the variables are reversed.

Q4 Drag *a* so it's exactly 9.00. What is *s*? Make *a* exactly 32.00. What is *s* now? If *a* were 76.00, between what two positive integers would *s* be? Explain.

5. In this case, what are the independent and dependent variables? Plot this new point as in step 3. Label the point *Q*.

Q5 Drag *a* to the left and to the right. How does *Q* behave? What happens when you drag *a* to the left of zero? Explain why this happens.

To plot a function, make sure nothing is selected and choose **Graph | Plot New Function.** Square root appears as **sqrt** on the function pop-up menu.

6. Plot the functions $f(x) = x^2$ and $g(x) = \sqrt{x}$.

Q6 How do these graphs compare to the existing traces?

So far you have plotted inverse functions algebraically, by changing which variable is the independent variable and which is the dependent. On pages 2 and 3 you'll look at the process geometrically.

7. On page 2 are point A and the measurements of its x- and y-coordinates. To form an inverse relation, you must interchange these coordinates, using y for x and x for y. Interchange the coordinates of A by plotting the point (y_A, x_A).

8. Turn on tracing for both points, and drag point A around the coordinate plane. Try to make some interesting shapes.

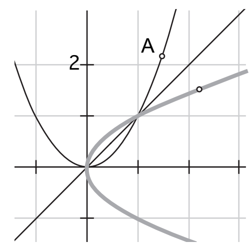
Q7 What kind of symmetry do your patterns show?

To erase all traces, choose **Display | Erase Traces.**

9. Erase the traces and drag A again. This time try to keep A and the plotted point together. This process will reveal the line of symmetry.

Q8 What is the equation of the line of symmetry?

Next you'll use the line of symmetry to construct the inverse of $y = x^2$.



To mark a mirror, select the line and choose **Transform | Mark Mirror.**

10. Page 3 shows the graph of $f(x) = x^2$ and the line of symmetry $y = x$. Mark the $y = x$ line as the mirror. Use the **Point** tool to put a point on the graph. Then reflect the point across the line.

To trace the point, select it and choose **Display | Trace Point.**

11. Drag the point on the graph. Turn on tracing for the reflected point, and drag again.

Q9 How does the reflected image point behave? Does it appear to trace out a function? (*Hint:* Try the vertical line test on it.)

Q10 Plot the function $g(x) = \sqrt{x}$. How does this function plot differ from the reflected image?

EXPLORE MORE

12. Page 4 shows the graph of $y = \sqrt{x}$ and the line of symmetry $y = x$. As before, put a point on the graph, reflect it across the line of symmetry, turn on tracing, and drag the point on the graph.

Q11 Plot the function $g(x) = x^2$. This time, how does the function plot differ from the reflected image?

Q12 Is either of the two reflections you constructed a function graph? Can you say that $f(x) = x^2$ and $g(x) = \sqrt{x}$ are inverse functions? If not, what restrictions must you specify in order to have inverse functions?

Objective: Students explore the square root function by generating data: First they vary the side length of a square and observe its area, and then they vary the area and observe the resulting side length. They investigate the relationship between these two functions using both algebraic and geometric methods. The activity ends by considering whether the inverse of x^2 is a function, and spurs students to think about the conditions under which inverse relations are also inverse functions.

Student Audience: Algebra 2

Prerequisites: None

Sketchpad Level: Intermediate

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Square Root Fns.gsp**) or Whole-Class Presentation (use **Square Root Fns Present.gsp**)

INVESTIGATE

Q1 When $s = 2$, $a = 4$. When $s = 5$, $a = 25$. When s is negative, a remains positive. As the value of s gets farther from zero, the value of a gets larger even faster.

Q2 In this model s is the value you change, so it's the independent variable, and a is the dependent variable.

Q3 The formula is $a = s^2$.

Q4 When $a = 9$, $s = 3$. When $a = 32$, $s = 5.66$. If a is 76, s must be between 8 and 9, because $8^2 = 64$, $9^2 = 81$, and 76 is between 64 and 81.

Q5 When you drag a to the right, Q follows a curving path in Quadrant I. When a is to the left of zero, point Q disappears, because the model does not allow a negative area.

Q6 The graphs of the two functions appear to match the traces that result from dragging the sliders.

Q7 The traces show reflection symmetry about a line that goes diagonally up and to the right.

Q8 The line of symmetry is $y = x$.

Q9 The reflected point traces out a full reflection of the x^2 function plot, in the shape of a parabola opening to the right. This trace does not define a function plot, because it fails the vertical line test.

Q10 The plot of $g(x)$ coincides with the trace in Quadrant I, where y is positive. However, the function plot stops at the origin and does not go below the x-axis, and so leaves out the lower branch of the parabola. The trace includes both the upper and lower branches of the parabola.

EXPLORE MORE

Q11 The reflected image of the square root function consists of the right-hand branch of a parabola. The plotted function $g(x) = x^2$ defines both branches of the parabola.

Q12 The reflection of $f(x) = x^2$ is not a function graph, because the inclusion of points like (4, 2) and (4, −2) makes it fail the vertical line test. The reflection of $y = \sqrt{x}$ is a function graph, consisting of the portion of $y = x^2$ for which $x \geq 0$.

If we restrict $f(x) = x^2$ to the domain $x \geq 0$, its inverse is a function: $g(x) = \sqrt{x}$. (This is called a *one-to-one function.*)

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Square Root Fns Present.gsp** to present this activity to the whole class.

Use this presentation to explore the square root function, to investigate its relationship to $y = x^2$, and to stimulate students to think about inverse relations and inverse functions.

1. Open **Square Root Fns Present.gsp.** This page shows in two different ways the relationship between the side and the area of a square.
2. Drag slider *s* and have students observe the changing values of *s* and *a*. Then show the plotted point *P* and continue dragging the slider to trace out the graph.

Q1 How can you tell which is the independent variable? Write a formula for *a* in terms of *s*. ($a = s^2$)

3. Drag slider *a* for the lower square and have students observe the changing values. Show the plotted point *Q* and continue dragging to trace out the graph.

Q2 How can you tell which is the independent variable? Write a formula for *s* in terms of *a*. ($s = \sqrt{a}$)

Q3 Ask students to describe each graph individually, and then to compare the two. What is similar about them? How are they related to each other? What are the important differences?

4. Press *Show x^2* and *Show sqrt(x)*.

Q4 How do these graphs compare to the existing traces?

So far you have plotted inverse functions algebraically, by changing which variable is the independent variable and which is the dependent. On page 2 you'll look at the process geometrically.

5. Page 2 shows the graph of $y = x^2$ and the line $y = x$. There's a point on the graph of $y = x^2$. To reflect this point across $y = x$, click *Show Point Reflection.*

Q5 Drag the point on the graph. Turn on tracing for the reflected point and drag again. How does the reflected image point behave?

Q6 Click *Show Function Reflection* to see the entire function reflected. How is it related to the reflected image?

Q7 Is the reflection the plot of a function? Does it pass the vertical line test?

Q8 How does the reflection compare to the graph of the function $f(x) = \sqrt{x}$?

Q9 On page 3, compare the reflection of $f(x) = \sqrt{x}$ to the graph of $y = x^2$.

Q10 Discuss with students how to modify the situation so that you have two functions, each of which is the inverse of the other. What sort of restrictions would be necessary?

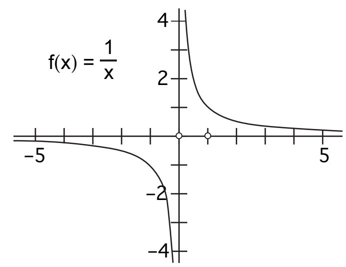
Rational Functions

A rational number gets its name because it is the ratio of two integers. Rational functions are named for similar reasons. A rational function is the ratio of two polynomial functions.

If $f(x)$ is a rational function, then $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomial functions. The polynomials can be of any degree, but in this investigation they will have degree one or zero.

THE RECIPROCAL FUNCTION

One simple example of a rational function is the reciprocal function $1/x$. This will be the first function in your investigation.



1. In a new sketch, define the coordinate axes. Adjust the scale so that the x-axis fits your screen between about -10 and 10.
2. Choose **Graph | Plot New Function.** Define the new function $f(x)$:

$$f(x) = \frac{1}{x}$$

Q1 Does $f(x)$ satisfy the definition of a rational function? Explain.

Q2 For what values of x is $f(x)$ zero? For what values is it undefined?

The graph $y = f(x)$ is a hyperbola. Like all hyperbolas, it has two *asymptotes*, lines that the curve approaches at the extremes.

Q3 What are the equations of the asymptotes of this curve? Asymptotes are lines, so your answer should be the equations of two lines.

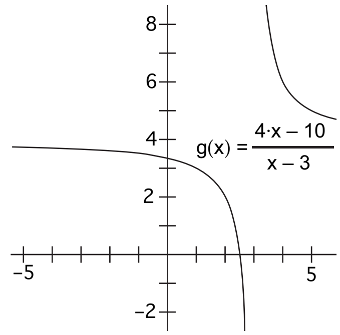
TRANSFORMATIONS OF THE RECIPROCAL FUNCTION

Use color to match each function with its graph. Select the function definition and its graph, and choose **Display | Color** to set a new color.

3. On the same grid, plot another rational function:

$$g(x) = \frac{4x - 10}{x - 3}$$

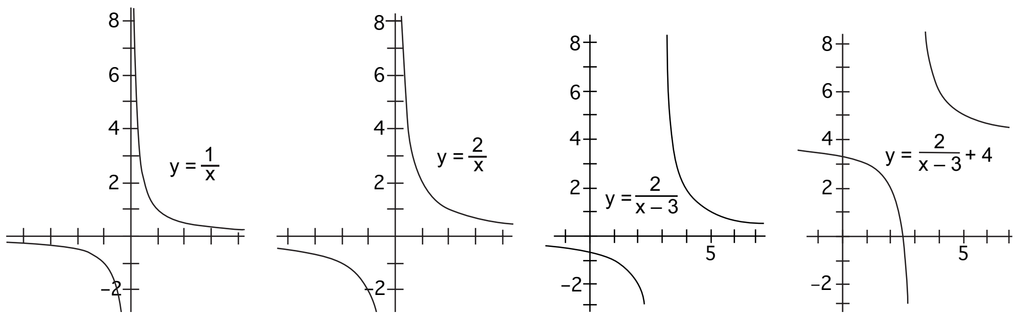
Q4 What are the equations of the asymptotes of the graph of $g(x)$?



Here's how it works. The function $g(x)$ is a fraction. Divide the denominator into the numerator, and leave a remainder.

$$g(x) = \frac{4x - 10}{x - 3} = \frac{4(x - 3) + 2}{x - 3} = \frac{2}{x - 3} + 4$$

When you view the function this way, you can see $g(x)$ in terms of the reciprocal function. Since $f(x) = 1/x$, it follows that $g(x) = 2f(x - 3) + 4$. Expressed as a transformation, this stretches the parent function, $f(x)$, vertically by a ratio of 2, and translates it right 3 units and up 4 units.



Original Stretched vertically Translated right Translated up

Q5 How does this transformation help explain the positions of the asymptotes of $g(x)$?

You can verify this by plotting a third function as a transformation of $f(x)$. Its graph should fall right on top of the graph of $g(x)$.

4. Plot $h(x)$ with the following definition:

$$h(x) = 2f(x - 3) + 4$$

Q6 Below are three new definitions to try for $g(x)$. In each case, express the function in terms of $f(x)$, as in the example above. Find the asymptotes of each graph. Check your work by plotting $g(x)$ and $h(x)$ in the sketch.

a. $g(x) = \dfrac{-3x - 12}{x + 5}$ b. $g(x) = \dfrac{4x - 17}{4x - 16}$ c. $g(x) = \dfrac{15x + 103}{5x + 35}$

Q7 Given the two asymptotes, it is possible to find any number of different rational functions that fit them. Define two different functions having the asymptotes $x = 6$ and $y = -4$. Start by showing the functions as transformations of $f(x)$, and then express them as ratios of polynomials.

Objective: Students define simple rational functions and show their graphs as transformations of the graph of the reciprocal function: $y = 1/x$.

Student Audience: Algebra 2

Prerequisites: This activity depends heavily on the concept of function transformation. Students should be comfortable with that before they begin.

Sketchpad Level: Intermediate. Students graph three functions, building the sketch from scratch.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Rational Functions Present.gsp**)

THE RECIPROCAL FUNCTION

Q1 Yes, this does fit the definition. Both 1 and x are polynomials. You may need to remind students that a polynomial may have only one term, and it does not necessarily have to have a variable.

Q2 The function $f(x)$ is not zero for any real x. It is undefined for $x = 0$.

Q3 The asymptotes are the x- and y-axes. Their equations are $x = 0$ and $y = 0$.

TRANSFORMATIONS OF THE RECIPROCAL FUNCTION

Q4 The asymptotes of $g(x)$ are the lines $x = 3$ and $y = 4$.

Q5 The parent function, $f(x) = 1/x$, has asymptotes at the x- and y-axes. The stretching transformation does not change either of these lines. The translations move the vertical asymptote right 3 units to $x = 3$ and the horizontal asymptote up 4 units to $y = 4$.

Q6

	Transformation Function	Asymptotes	
a.	$3f(x + 5) - 3$	$x = -5$	$y = -3$
b.	$-\frac{1}{4}f(x - 4) + 1$	$x = 4$	$y = 1$
c.	$-\frac{2}{5}f(x + 7) + 3$	$x = -7$	$y = 3$

Q7 Answers will vary. For any pair of asymptotes, $x = h$ and $y = k$, it is possible to write the function as a transformation in this form:

$$g(x) = a \cdot f(x - h) + k$$

This transformation stretches the graph vertically by a factor of a and translates it to new center point (h, k). In this instance $h = 6$ and $k = -4$. Since a, the ratio of the stretch, can be any real number, there is no limit to the number of unique rational functions that fit these asymptotes.

To express the function as a ratio of polynomials, substitute 6 for h and -4 for k, and use $f(x) = 1/x$. Then combine the parts into a single fraction.

$$g(x) = a \cdot f(x - 6) - 4 = \frac{a}{x - 6} - 4 = \frac{-4x + 24 + a}{x - 6}$$

Any function in this form will have the required asymptotes.

EXTENSION

Throughout this activity the stretching transformations are called *vertical stretches.* In fact, because of a special property of the reciprocal function, you could just as well call them *horizontal stretches.*

Again let $f(x) = 1/x$. To stretch the graph vertically by ratio a $(a \neq 0)$, you would graph the curve $y = a \cdot f(x)$.

$$a \cdot f(x) = \frac{a}{x} = \frac{1}{\frac{1}{a}x} = f\left(\frac{1}{a}x\right)$$

This shows that a vertical stretch by ratio a is equivalent to a horizontal stretch by the same ratio. Although you could say that the graph is stretched vertically or horizontally by ratio a, it would not be correct to say that it is stretched vertically *and* horizontally by ratio a. That would be a dilation by ratio a, which this is not.

This activity addresses a class of rational functions in this form:

$$g(x) = \frac{cx + d}{ex + f}, \quad \text{where } e \neq 0$$

Understanding the properties of the function and the shape of its graph becomes easier when you rewrite the function:

$$g(x) = \frac{a}{x - h} + k$$

In this form you can create the function by applying three transformations (corresponding to a, h, and k) to the reciprocal function $f(x) = 1/x$.

1. Open **Rational Functions Present.gsp.** The curve on the screen is the graph of a rational function.
2. Guide students through these steps to change the form of the function:

$$\frac{4x - 10}{x - 3} = \frac{4(x - 3) + 2}{x - 3} = \frac{2}{x - 3} + 4$$

3. If $f(x) = 1/x$, then this function is $2f(x - 3) + 4$. You should be able to get the same graph by putting the parent function (the reciprocal curve $y = 1/x$) through three transformations. Stretch it vertically by a ratio of 2, translate it right 3 units, and translate it up 4 units.
4. Press the *Show Parent Function* button. This shows the graph of $y = 1/x$.
5. Show the three transformations by pressing the corresponding buttons (*Stretch, Horizontal Translation,* and *Vertical Translation*) in order, pausing at each step to check for understanding. You will also be able to see the transformations of the asymptotes. Press *Reset* when you finish.

Do not change the order of the transformations.

6. Above and below the equation in the upper-left corner are four parameters corresponding to the parameters in the rational function. Edit the parameters in order to change the equation. Try the equations below. Have the class change the form of the function as you did in the first example. (The solutions are below the problems.) Repeat step 5 to see the transformations.

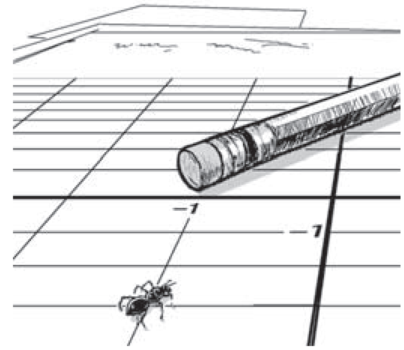
To edit a parameter, either double-click it or select it and press the plus (**+**) or minus (**–**) key.

a. $y = \dfrac{-3x - 12}{x + 5} = \dfrac{3}{x + 5} - 3$

b. $y = \dfrac{4x - 17}{4x - 16} = \dfrac{-0.25}{x - 4} + 1$

c. $y = \dfrac{15x + 103}{5x + 35} = \dfrac{-0.4}{x + 7} + 3$

Modeling Linear Motion: An Ant’s Progress



You notice an ant walking slowly across a piece of graph paper on your desk. You start your stopwatch the moment the ant is at point $(-4, -5)$; one minute later, it has reached point $(-1, -3)$. You find yourself wondering when the ant, if it keeps up this pace, will cross the y-axis and, after that, the x-axis.

Spend some time pondering this situation. For now, focus on understanding and picturing the situation rather than finding exact solutions. If you do come up with possible answers, write them down.

INVESTIGATE

Now that you have an understanding of the situation, let’s look at how algebra and Sketchpad can help you explore the ant’s walk.

Consider the following two equations, which together give the ant’s location at any time *t:*

Equations like these are called *parametric equations,* and the variable *t* is called a *parameter.*

$$x = -4 + 3t \quad \text{and} \quad y = -5 + 2t,$$

where t is the number of minutes the ant has walked.

Q1 Use the equations above and the given values of t to fill in the table. To do this, substitute the t-values into the equations to find the ant’s (x, y) locations at the given times.

For example, for $t = 0$:

$$x = -4 + 3(0) = -4$$

$$y = -5 + 2(0) = -5$$

which confirms that the ant is at point $(-4, -5)$ at time $t = 0$.

t	0	1	2	3	4
x	-4				
y	-5				

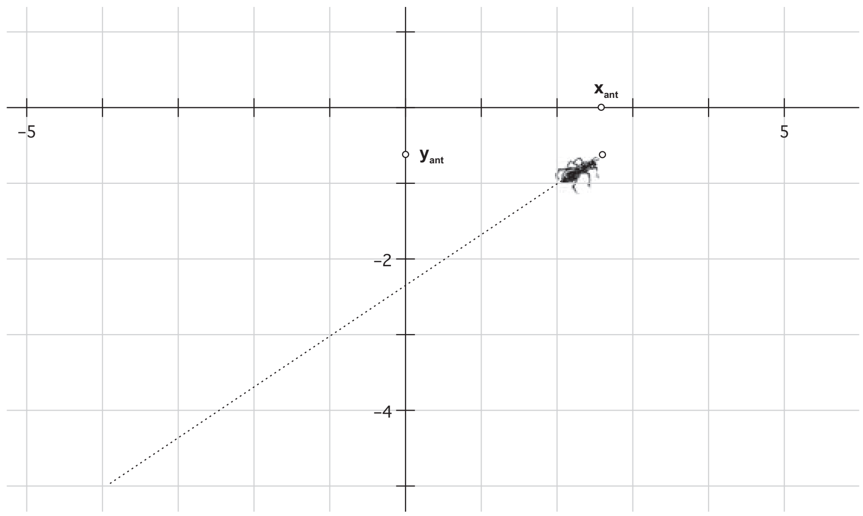
Q2 Plot each (x, y) pair from Q1 on a coordinate grid (either in Sketchpad or on your paper) to show where the ant is after 0, 1, 2, 3, and 4 minutes.

Q3 What would the two parametric equations be if the ant started at point $(-2, 0)$ and reached point $(0, 1)$ after one minute?

SKETCH

1. Open **Linear Motion.gsp.**

You'll see a coordinate grid with a red line representing time, the parametric equations, and, of course, an ant.



2. Familiarize yourself with the sketch by moving the ant along its path (by dragging marker *t*) and also by pressing the buttons in the sketch. Compare the ant's locations in the sketch with the positions you plotted by hand.

When the ant crosses the *x*-axis, the *y*-value must be zero. Substitute zero in the parametric equation for *y*.

Q4 Use the parametric equations themselves to determine the exact times when the ant crosses the *x*- and *y*-axes. Use the sketch to check your answers.

Q5 What are the *x*- and *y*-intercepts of the line the ant is following?

Press and hold the **Straightedge** tool to choose the **Line** tool from the pop-up menu.

Q6 With the trace showing, draw a line over it as accurately as you can. Select the line and measure its equation. The measurement appears in slope-intercept form. How are the slope and intercept related to the two parametric equations?

To edit the parametric equations, double-click them. To enter *t* into the Calculator dialog box that appears, click on its measurement in the sketch.

Q7 Consider the following motion: The ant starts at (5, 1) and heads toward (−7, 19). Edit the parametric equations to model this situation, and test the model by pressing the action buttons.

Q8 How would the equations in Q6 change if you insisted that the ant travel the same speed as at the beginning of the activity? Edit the sketch to model this situation, test the model by pressing the action buttons, and write your equations on your paper.

Modeling Linear Motion: An Ant's Progress

Objective: Students model linear motion using parametric equations.

Student Audience: Algebra 2

Prerequisites: None

Sketchpad Level: Easy

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Linear Motion.gsp**) or Whole-Class Presentation (use **Linear Motion Present.gsp**)

Although this activity covers an advanced topic, parametric equations of lines, it is an accessible and fun activity for students to do. It's especially valuable for the connections between the parametric equations and the slope-intercept form of lines.

INVESTIGATE

Q1

t	0	1	2	3	4
x	−4	−1	2	5	8
y	−5	−3	−1	1	3

Q2 Students should plot the following points: $(-4, -5)$, $(-1, -3)$, $(2, -1)$, $(5, 1)$, and $(8, 3)$

Q3 $x = -2 + 2t$ and $y = 0 + 1t$ (or $y = t$)

SKETCH

Q4 When the ant crosses the y-axis, $x = 0$.

$$-4 + 3t = 0$$

$$t = \frac{4}{3} \text{ (1 minute, 20 seconds)}$$

When it crosses the x-axis, $y = 0$.

$$-5 + 2t = 0$$

$$t = \frac{5}{2} \text{ (2 minute, 30 seconds)}$$

Q5 You can find the intercepts from the answers to Q4. The ant crosses the y-axis when $t = 4/3$.

$$y = -5 + 2\left(\frac{4}{3}\right) = -\frac{7}{3}$$

It crosses the x-axis when $t = 5/2$.

$$x = -4 + 3\left(\frac{5}{2}\right) = \frac{7}{2}$$

Q6 The equation is approximately $y = 0.67x - 2.33$. Written using fractions, this is $y = 2/3x - 7/3$. (Depending on how accurately students have drawn their lines, their results for the slope and y-intercept may differ by a few hundredths.) The slope (0.67, or 2/3) is the ratio of the coefficients of t in the equations for y and x. The y-intercept is the value of y when x is zero (from Q4, at $t = 4/3$). So the y-intercept is

$$y = -5 + 2t = -5 + 2(4/3) = -5 + 8/3 = -7/3$$

This matches the measured value of −2.33.

Q7 Since the speed of the ant is not given, there are many correct answers, but these are the most likely:

$$x = 5 - 12t, \quad y = 1 + 18t$$

The ant must start at (5, 1), and the path must have a slope of −3/2. The slope is the ratio of the t coefficients.

Q8 In the first situation, the ant was going right 3 units and up 2 units every minute. If it goes left 2 units and up 3 units per minute, it will be traveling at the same speed, and the slope of its path will satisfy the conditions of this new problem.

$$x = 5 - 2t, \quad y = 1 + 3t$$

Parametric equations use x and y as separate functions of a third variable, t. This idea is a considerable leap. Typically, you will introduce t as time. But with this presentation you have an advantage over chalkboard lessons. Here the variables actually do change over time.

1. Open **Linear Motion Present.gsp.** Describe the scenario:

   You have a sheet of graph paper on your table, and you see an ant slowly walking across the grid. Unlike most ants, this one appears to know where it is going. It is walking in a straight line and at a steady pace.

The actual speed of the animation will be much faster. You will need to emphasize that the time units are minutes.

2. Press *Show Ant's Motion* and ask students to observe the ant's journey.
3. Explain that you are going to trace the first minute of the trip. Point out the measurement t, which represents the time in minutes. Press in order *Reset, Show Traces,* and *Advance 1 minute.*

Q1 What were the coordinates of the ant at time zero? $(-4, -5)$ What are its coordinates at time 1? $(-1, -3)$

Q2 Tell students to think about the x-coordinate only. It started at -4, and after 1 minute, it was -1. What will it be after 2 minutes? (2) What will it be after 3 minutes? (5)

Q3 So the x-coordinate of the ant starts at -4, and advances 3 units every minute. How can students express x in terms of t? ($x = -4 + 3t$)

Q4 Using the same reasoning, what is y in terms of t? ($y = -5 + 2t$)

4. Press *Show Equations* to show the same equations derived above. Explain the concept of parametric equations. Rather than y being a function of x, you have both x and y as functions of parameter t.

Q5 Press *Show Ant's Motion,* and let it run long enough that students can clearly see the linear path. What is the slope of the path? How can they derive it from the parametric equations? (The slope is 2/3, which is the ratio of the coefficients of the t terms in the equations.)

After getting answers to Q6 and Q7 (right or wrong), double-click the equations to make the changes.

Q6 Ask, "How can you change the equations to make the ant go twice as fast without changing its path?" (Multiply both of the t terms by two. $x = -4 + 6t$, $y = -5 + 4t$)

Q7 A different ant starts at $(-6, 2)$, and after 1 minute it is at $(-1, 0)$. What parametric equations model its motion? ($x = -6 + 5t$, $y = 2 - 2t$)