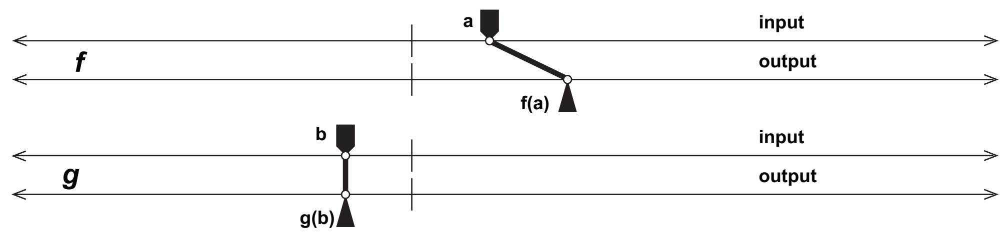
1

Functions

Introducing Dynagraphs

A *function* is a mechanism that gives you one specific output value for any value that you put in.

How many ways are there to represent a function? You've probably encountered various representations of functions, using tables, graphs, or equations. In this activity you'll explore a new way of representing functions: *dynagraphs.*

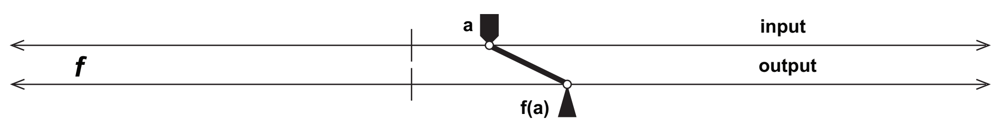


SKETCH AND INVESTIGATE

Q1 Spend a few moments reviewing with your group or on your own what a function is. Based on what you already know, how would you describe functions to someone who isn't familiar with them?

Each dynagraph has an input axis with an input marker and an output axis with an output marker. There is also a tick mark in the middle of each axis.

1. Open **Introducing Dynagraphs.gsp.** You'll see four dynagraphs labeled *f, g, h,* and *j,* each in a different color.



2. The input marker for dynagraph *f* is labeled *a.* To get an idea of how dynagraphs work, use the **Arrow** tool to drag this input marker.

Q2 Based on your understanding of functions, does this dynagraph represent a function? Explain.

Next you'll explore and describe in detail each of the dynagraphs on this page.

Here's a description of the *f* dynagraph:

This description is in terms of the position and motion of the input and output markers. The description does not use numbers or formulas because there are no numbers or formulas on the dynagraphs.

When the input marker is at the tick mark, the output marker is also at the tick mark. When the input marker is not at the tick mark, the output is always on the same side of the tick mark as the input. The output is always farther away from the tick mark than the input; it seems to be about twice as far away. When the input is dragged steadily from left to right, the output also moves steadily in the same direction, only faster.

Q3 Drag the input markers for dynagraphs *g*, *h*, and *j*, and then write detailed descriptions of these functions. Imagine you're describing the dynagraphs to someone who can't see them.

NUMBERS, NUMBERS, NUMBERS

You may have thought that it would be convenient to have number lines as the dynagraph axes to make it easier to give a precise description of the behavior of each function. With only a single tick mark, it's impossible to assign numbers to positions, such as "an input of 3 gives an output of 5."

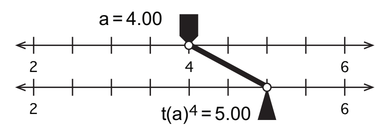
In this section you'll explore four new dynagraphs, first without numbers, then with.

3. On page 2 are the new dynagraphs. Explore each by dragging its input marker.

Q4 Write a description of each function, just as in the previous section.

4. Press the *Show Number Lines* button. The dynagraph axes appear as number lines.

5. Drag t's input control to 4.



The arrow points to an output of 5, as shown here. Using function notation, you can write $t(4) = 5$, which is read "t of four equals five."

Q5 Solve for these unknowns. Be sure to use the correct function dynagraph for each one. Write each answer using function notation.

Hint: The answers to all but two of these questions are single numbers. One answer is "undefined," and one consists of several numbers.

a. $t(1) =$ | b. $t(5) =$

c. $t(x) = -5, x =$ | d. $u(-1) = g, g =$

e. $u(3) = p, p =$ | f. $u(m) = 6, m =$

g. $v(4) =$ | h. $v(-4) =$

i. $v(r) = 3, r =$ | j. $w(2) = z, z =$

k. $w(4) = s, s =$ | l. $w(a) = 0, a =$

EXPLORE MORE

A function has an *absolute maximum* if there is a largest output value—one the function can reach but can never exceed. Similarly, a function has an *absolute minimum* if there is a smallest output value—one the function can reach but can never go below.

Q6 Of the eight functions in the sketch, which functions have an absolute maximum or an absolute minimum? What are these maximum/minimum output values, and for what input values do they occur?

Introducing Dynagraphs

Objective: Students explore dynagraphs, an alternative to Cartesian graphs, to develop a feel for various types of functional relationships.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should have some idea of what is meant by a function.

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 30–40 minutes. Be sure to give students enough time to write detailed and precise descriptions of the dynagraphs (Q3 and Q4). To reduce the amount of time required, students could skip Q4 or the Explore More section.

Setting: Paired/Individual Activity (use **Introducing Dynagraphs.gsp**) or Whole-Class Presentation (use **Introducing Dynagraphs Present.gsp**)

The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O'Keefe in their study "Dynamic Representation and the Development of a Process Understanding of Functions" published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation.

The motivation for developing and using dynagraphs comes from the often-noted difficulty students have in seeing the graphs of functions as dynamic representations of functional relationships between two quantities and not just as static pictures. By decoupling the input and output axes, and having a segment connect points on parallel axes, students are better able to see the input-output machine view of functions expressed graphically. Being able to drag the input marker gives students the further advantage of actually varying the independent variable and seeing the function as a *dynamic* relationship between input and output.

Dynagraphs can serve as a bridge between the input-output machine model with which students are often introduced to functions and function graphs in the Cartesian plane.

SKETCH AND INVESTIGATE

Q1 Answers will vary, but should basically describe functions as consistent input-output machines. In other words, they are relations or mappings between input values and output values such that any valid input value maps to a single output value.

Q2 The dynagraphs do represent functions because they map input values to output values and they are consistent—a particular input value will always point to the same output value.

Q3 Answers will vary, but should not involve numbers or formulas. Good answers will in general include dynamic descriptions ("As the input is dragged steadily from left to right, the output . . . ") and note any symmetries present.

NUMBERS, NUMBERS, NUMBERS

Q4 See Q3.

Q5

a. $t(1) = -1$	b. $t(5) = 7$
c. $x = -1$	d. $g = -6$
e. $p = -6$	f. $m = \ldots, -7, -3, 1, 5, \ldots$
g. $v(4) = 2$	h. $v(-4)$ is undefined
i. $r = 9$	j. $z = 1$
k. $s = 3$	l. $a = 2.5$

EXPLORE MORE

Q6 Function j has an absolute minimum of 0 at 0. Function u has an absolute maximum of 6 at $(\ldots, -7, -3, 1, 5, \ldots)$, and has an absolute minimum of -6 at $(\ldots, -5, -1, 3, 7, \ldots)$. Function v has an absolute minimum of 0 at 0.

WHOLE-CLASS PRESENTATION

Use **Introducing Dynagraphs Present.gsp** to explore this highly dynamic visual representation of functions with your students. Dynagraphs differ from Cartesian graphs in that you can make the variables really vary, so emphasize the variation in the presentation by using the Animation buttons and leaving variables moving on the screen.

This whole-class presentation allows students to gain a dynamic perspective on the notion of function and emphasizes the way in which the variables really vary.

SKETCH AND INVESTIGATE

Q1 Begin by asking students to describe a function in their own words. Get responses from several students, and encourage a diversity of descriptions. Consider forming small groups of two or three students and asking each group to create its own written description, suitable for explaining functions to someone who isn't familiar with them.

1. Open **Introducing Dynagraphs Present.gsp.** Four dynagraphs appear, each in a different color.

2. Explain that the input and output markers represent the variables and that this model allows you to vary the variables by dragging the markers. Use the **Arrow** tool to drag input marker *a*. After dragging it a bit, use the *Animate a* button to leave it in motion.

Q2 Ask students whether the behavior they observe represents a function, based on their description of what a function is. Solicit different explanations from as many students as possible.

Q3 Ask students to describe the behavior of this first function. They will want to call the tick mark "zero" or "the origin," and they will want to describe movement to the left or right as "increasing" or "decreasing." These characterizations are based on numbers; resist them, and instead encourage students to describe the behavior in terms of position, movement, and symmetry. Consider asking students whether the function has a "fixed point"—a state in which the input marker and the output marker are at exactly the same position.

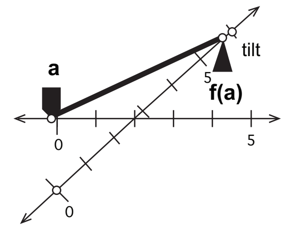
Leave each function in motion while you drag and discuss the remaining functions.

Q4 Drag the input marker for the second function and have students observe. Leave it in motion while students describe the behavior. (Students often want to call this function a "constant function." Rather than describing this answer as wrong, ask them whether it's the output that is constant or whether there is something else about this function that they view as being "constant.")

Q5 Use the *Animate c* button to put the third function's input marker into motion. (Students will often laugh at this function, and you may want to ask them how often they have laughed at a mathematical function.) Have them describe this function in detail. They may want to give it a name.

3. The remaining pages show dynagraphs with numbers added to the axes. Have students answer the questions on each page, and then allow them to see the algebraic formulas underlying the behavior of the dynagraphs.

From Dynagraphs to Cartesian Graphs



Dynagraphs make it easy to change the input of a function and see how each input produces a corresponding output. This strength is also a weakness, because you can see only a single pair of input-output values at any time.

In this activity you'll change a dynagraph so that you can keep track of many input and output values at the same time.

FUNCTION MATCHMAKING

Start out with some "function matchmaking"—you'll match several dynagraphs to their corresponding algebraic equations.

The dynagraphs don't show any numbers, so you'll have to figure out each match by observing the output marker as you drag the input marker.

Q1 Open **Dyna To Cartesian.gsp.** The five dynagraphs on page 1 (labeled "Matching") correspond to the equations below. Pair each dynagraph with an equation and explain how you made the match.

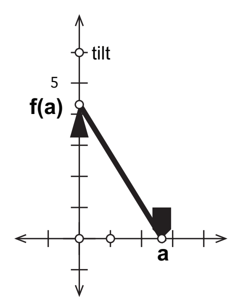
a. $y = x$ b. $y = -x$ c. $y = 2x$

d. $y = x^2$ e. $y = \dfrac{1}{x - 1} + 1$

FROM DYNAGRAPHS TO CARTESIAN GRAPHS

1. On page 2 of **Dyna To Cartesian.gsp,** you'll see a dynagraph for the function $f(x) = 2x - 1$. Drag the input marker to familiarize yourself with this function.
2. Drag the point labeled *tilt* so that the output axis is at an angle to the input axis. Drag the input marker again. Do this for a few different angles of the output axis. (You can even turn it upside down!)

Q2 When you tilt the output axis, what changes and what remains the same?



3. Press the *Make Cartesian* button and watch as the two axes of your dynagraph "morph" into the familiar x- and y-axes. Drag the input once more to convince yourself that you're still dealing with the same dynagraph, only tilted.
4. Press the *Show Perpendiculars* button to show lines through a and $f(a)$ perpendicular to the two axes.

Click on the intersection to construct the point of intersection.

5. Construct the intersection of the two perpendiculars.
6. With the new point selected, measure its coordinates by choosing **Measure | Coordinates**.

Q3 Drag the input marker. What does the x-coordinate of the new point correspond to on the dynagraph? What does the y-coordinate correspond to?

7. Deselect all objects by clicking in blank space. Select the new point and choose **Display | Trace Intersection**. Now drag or animate the input marker and watch as P traces out the graph of $f(x)$.

Q4 Describe the shape of the graph traced by the intersection point. Why does this shape make sense given the behavior of the dynagraph?

SIMULTANEOUS REPRESENTATION

8. Go to page 3 of **Dyna To Cartesian.gsp.**

You'll see a dynagraph and a Cartesian graph, both modeling $f(x) = 2x - 1$. Drag the input marker on the dynagraph and watch both models change simultaneously.

Q5 A classmate says, "One Cartesian point contains the same information as two dynagraph points." Explain what she means.

Double-click the function equation f(x) = 2x – 1 to edit it.

9. Explore each of the following functions on the combined dynagraph/Cartesian graph. Enter the function, then drag the input marker slowly from left to right. Observe what happens to the point on the Cartesian graph as you drag.

$$f(x) = 3 \qquad f(x) = x \qquad f(x) = -x$$
$$f(x) = x^2 \qquad f(x) = -x^2 \qquad f(x) = 5x$$

Q6 Fill in the blanks.

When the input and output markers both move right, the Cartesian point moves ____________.

When the input marker moves right and the output marker moves left, the Cartesian point moves ____________.

Q7 How does the Cartesian graph of $f(x) = 5x$ compare to that of $f(x) = x$? How does this relate to the difference between their dynagraphs?

Q8 Compare dynagraphs and Cartesian graphs. In what ways do you think dynagraphs are better for representing functions? In what ways do you think Cartesian graphs are better?

Objective: Students make connections between symbolic and dynagraph representations of functions by matching dynagraphs with equations, and then generate a Cartesian representation by geometrically manipulating a dynagraph.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should already have been introduced to dynagraphs.

Sketchpad Level: Easy. Students manipulate a pre-made sketch and add to it minimally.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Dyna To Cartesian.gsp**) or Whole-Class Presentation (use **Dyna To Cartesian Present.gsp**)

The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O'Keefe in their study "Dynamic Representation and the Development of a Process Understanding of Functions" published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation.

FUNCTION MATCHMAKING

Q1 Explanations will vary, but should involve the relative direction and speed of the input and output. Students can animate the input marker to make it easier to judge relative speed. Explanations may also refer to range restrictions (in d) and the existence of an asymptote (in e). Here are the matches:

a. 4 b. 2 c. 5

d. 1 e. 3

FROM DYNAGRAPHS TO CARTESIAN GRAPHS

Q2 Answers will vary. When the output axis is tilted, the two axes intersect, and there's a change in the length and orientation of the segment connecting the two dynagraph markers. Even as these things change, the output value remains the same for any given input value, so the two markers continue to represent the same function. If $f(a) = b$ before tilting, $f(a) = b$ after tilting.

4. Consider having students construct the perpendiculars on their own instead of pressing the button.

Q3 The x-coordinate of the Cartesian point is a (the input value of the dynagraph). Similarly, the y-coordinate of the new Cartesian point is $f(a)$ (the output value of the dynagraph).

7. After generating the trace, students can use **Construct | Locus** to create a permanent image. But it's important that they trace first, to get a visual image of the way in which the dynagraph traces out the Cartesian graph over time.

Q4 The graph is a line with a positive slope. The fact that it's a line corresponds to the fact that as the input marker on the dynagraph is dragged at a constant speed, the output marker moves at a constant speed. Students may make additional observations: How far the input marker goes in a given unit of time corresponds to *run,* and how far the output marker goes corresponds to *rise.* The fact that the line has a positive slope corresponds to the fact that the two markers always go the same direction (see Q6). The fact that the slope of the line is greater than 1 corresponds to the fact that the output marker moves faster than the input marker. (A slope of 1 corresponds to markers moving at the same speed; a slope between 0 and 1 corresponds to a slower output marker.)

SIMULTANEOUS REPRESENTATION

Q5 A dynagraph point contains one piece of information: its position on its axis. Thus two dynagraph points are required to show both input and output values for a function. A Cartesian point contains two pieces of information—its x-coordinate and its y-coordinate—so the Cartesian point can represent both the input and the output values at the same time.

Q6 When the input and output markers both move right, the Cartesian point moves up and to the right.

When the input marker moves right and the output marker moves left, the Cartesian point moves down and to the right.

Q7 The Cartesian graph of $f(x) = 5x$ is *steeper* than the graph of $f(x) = x$. This corresponds to the fact that the output marker moves faster for the dynagraph of $f(x) = 5x$ than it does for that of $f(x) = x$.

Q8 Student answers will vary, but should describe one advantage of each representation. Students may say that it's easier to distinguish input values from output values on a dynagraph, and that the dynagraph allows them to actually vary the variable. The dynagraph relates more closely to the fundamental definition of a function: For any given input value, it produces one single output value. The Cartesian graph makes the global picture more accessible: At a glance you can see the characteristics of the entire function for all input values, with no need to drag or animate anything. Finally, the dynagraph shows clearly where each point on the Cartesian graph comes from.

WHOLE-CLASS PRESENTATION

In this presentation, students make connections between the behavior of a dynagraph that represents a function and an equation representing the same function. They then see how modifying the dynagraph representation produces a Cartesian representation, and they view different functions in both representations simultaneously.

1. Open **Dyna To Cartesian Present.gsp.** Press *Animate* to put the top dynagraph into motion.

Q1 Have students guess which of the five equations on the bottom of the screen corresponds to the top dynagraph. Ask them to give arguments for their guesses, and encourage observations that relate the direction or speed of the dynagraph markers to the equations. Repeat for all five functions.

2. Page 2 contains a single dynagraph. Put it in motion.

Q2 Ask students what they think will happen if you drag the point labeled *tilt.*

Q3 With the input marker still moving, drag point *tilt* so that the bottom axis is only slightly off horizontal. Ask students to observe what has changed about the behavior of the dynagraph and what has stayed the same. Try this with several different positions of point *tilt.* The class should agree that although the direction of the output marker's motion has changed, the motion of the marker along its axis, and its numeric value, remain unchanged.

3. With the input marker in motion, press the *Make Cartesian* button.

Q4 Ask students to report what they observe about the angle of the output axis and about the origin of the output axis.

4. With the input marker in motion, press the *Show Perpendiculars* button and ask students to observe the relationship between the dashed lines and the values of a and $f(a)$.
5. Show the intersection.

Q5 Ask students to observe the path of the intersection: its shape, its location on the screen, and so forth.

6. Turn tracing on so that students can check their answers.
7. Consider using this page to look at one or two different functions, such as $f(x) = x^2$ or $f(x) = |x|$.
8. Page 3 contains both dynagraph and Cartesian representations. Animate the input marker a.

Q6 Ask students what relationships they can detect about the connections between the two models. Students may make observations about the behavior of the markers on the four axes, or about the relation between the moving point labeled $(a, f(a))$ and the motion of the dynagraph.

9. Edit the equation to change the function being displayed, and ask students how looking at a different function changes their observations about the connections.

Finish with these three questions:

Q7 Ask students to fill in the blanks:

When the input and output markers both move right, the Cartesian point moves ______________.

When the input marker moves right and the output marker moves left, the Cartesian point moves ______________.

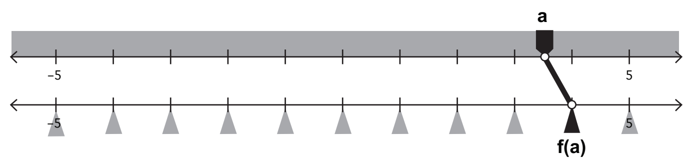
Q8 How does the Cartesian graph of $f(x) = 5x$ compare to that of $f(x) = x$? How does this relate to the difference between their dynagraphs?

Q9 Compare dynagraphs and Cartesian graphs. In what ways do you think dynagraphs are better for representing functions? In what ways do you think Cartesian graphs are better?

Domain and Range

You can't put a television in a blender, and you wouldn't expect an elephant to come out of a gasoline pump. In math terms, a television isn't an *allowable input* for a blender; it's not part of a blender's *domain.* And an elephant isn't a *possible output* of a gasoline pump; it's not part of a gas pump's *range.*

Similarly, functions have certain numbers that are and aren't allowed as inputs, and other numbers that are and aren't possible as outputs. In this activity you'll explore these notions using both dynagraphs and Cartesian graphs.



SKETCH AND INVESTIGATE

The *range* of a function is the set of possible outputs from that function. Let's see how dynagraphs can make this idea clearer. You'll start by exploring the range of everyone's favorite dynagraph: the "blue hopper."

1. Open **Domain Range.gsp.** Drag the input marker to observe the behavior of the function $f(x) = \text{round}(x)$. The input marker leaves a trace.

You can drag the input marker to any value you want, so we say "the domain of f is all real numbers."

To turn on tracing, select the output marker and choose **Display | Trace Triangle.**

2. Turn on tracing for the output marker. Then drag the input marker back and forth again.

The output marker leaves a trace of where it's been. These traces point to every integer but never to any other values, so we say "the range of f is all integers."

The range of each function on page 2 will either be "all real numbers" or an inequality such as $f(x) \geq 5$.

Q1 On page 2, use the same method to find the range of each of the four functions.

When you use technology, it's very important to think about the limitations of that technology. You'll see that the method used above can be misleading in certain situations, and you'll then learn a more reliable method.

3. On page 3, turn on tracing for the output marker, then drag the input marker, as in step 2. For greater control, use the right and left arrow keys on your keyboard to drag one pixel (screen unit) at a time.

Q2 What does the range of $k(x) = 20 \cdot x$ *appear* to be? Explain why this answer is actually wrong. Why do you think this happens?

4. Select the input marker and choose **Display | Animate Pentagon.** Repeatedly press the *Decrease Speed* button (the down arrow on the Motion Controller) until it's clear that the range of this function really is all real numbers. Go back to pages 1 and 2 of the sketch, and convince yourself that your answers there were correct.

For most functions, the domain is "all real numbers," meaning that any input produces some output. Sometimes, however, a domain might be *restricted* to something such as "all integers" or "$x > 3$."

Q3 On page 4, drag v's input marker back and forth. What is v's domain? (In other words, where can you drag the input marker and still see the output marker?) What is v's range?

Q4 Based on its equation, why are some numbers not part of v's domain?

Q5 The domain of w is all real numbers except for one particular value. The range of w is also all real numbers except for one particular value (a different value). What is the one value not in the domain of w? What is the one value not in the range of w?

DOMAIN AND RANGE ON CARTESIAN GRAPHS

Let's transfer this knowledge to Cartesian graphs.

5. Page 5 shows a rounding function. Drag the input marker and think about how you can tell domain and range on a Cartesian graph.

To edit the function, double-click its equation and then enter the new expression.

6. On page 6, explore the function $f(x) = 2x$. Then edit the function and explore the functions listed here, again thinking about how to tell domain and range on a Cartesian graph.

$$g(x) = \frac{x}{2} \qquad h(x) = x^2 \qquad j(x) = 2x + 1 \qquad k(x) = 2\sin\left(\frac{\pi x}{2}\right)$$

Q6 How can you tell the range of a function just by looking at its Cartesian graph? How about its domain?

EXPLORE MORE

Q7 On page 1, change f's equation so that its range is all even numbers. Then change it again to make it all odd numbers. Record the equations you used.

Q8 On page 4, change v so that both its domain and its range include only numbers less than or equal to 0. Change w so that its domain is all real numbers except 0 and its range is all real numbers except 2. Record the equations you used.

Objective: Students explore the domain and range of various functions using dynagraphs and Cartesian graphs. They drag input variables and observe the behavior of the input and output of the functions, with special attention to functions having restricted domain and/or range. Finally, they design functions to have particular domain and range restrictions.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should already have been introduced to dynagraphs in a previous activity such as Introducing Dynagraphs. Some prior introduction to domain and range is also very helpful for most students.

Sketchpad Level: Easy. Students manipulate a pre-made sketch and modify it only minimally.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Domain Range.gsp**) or Whole-Class Presentation (use **Domain Range Present.gsp**)

Related Activities: Introducing Dynagraphs, From Dynagraphs to Cartesian Graphs, Function Composition with Dynagraphs

The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O'Keefe in their study "Dynamic Representation and the Development of a Process Understanding of Functions" published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation. See the Activity Notes from the activity Introducing Dynagraphs for a more thorough discussion of dynagraphs in general.

It's a good idea to have an initial class discussion about domain and range before starting the activity, and it's very important to have a class discussion after students finish the activity.

One or the other of these discussions should focus on why some numbers are not allowed as inputs for particular functions. Ask students to give examples of functions with restricted domains. One category of answers is covered in the activity—functions such as square root functions and rational functions that have undefined outputs for certain inputs. Another category, not covered in the activity, is functions that model real-life or geometric situations. For example, the function $f(n) = 0.89n$ might represent the total cost of buying n apples each costing \$0.89. But it doesn't make sense to consider $n = -2$ or $n = 3.71$ here. Only positive whole numbers are part of this domain.

A good way to encourage exploration is to ask students to modify the various functions and observe the results. On page 1, you could ask students how the range would change if they changed the function to $f(x) = \text{round}(x) + 1$ (*answer:* it wouldn't). You could then ask for what values of k would the function $f(x) = \text{round}(x) + k$ have a range different from that of the original function (*answer:* non-integer values). You could follow this up by exploring what happens when you multiply round(x) by a constant. You could extend the activity in similar ways on the other pages.

SKETCH AND INVESTIGATE

Q1 $g(x)$: all real numbers
$h(x)$: $h(x) \geq 0$
$j(x)$: all real numbers
$k(x)$: $-2 \leq k(x) \leq 2$

Q2 The range appears to be all integers, as with $f(x) = \text{round}(x)$. This is clearly wrong since many inputs result in non-integer outputs. For example, $g(0.01) = 0.2$.

The output marker lands only on integers because the input marker can't really be dragged continuously. You can only move it by one pixel (screen unit) at a time. This dynagraph's scale is set to 1 pixel = 0.05 units. Moving the input marker by a single pixel moves the output marker by 20 times as much, which is equivalent to a full unit. You can change the scale by pressing the *Show Scale* button and adjusting the slider.

In the next step, students are encouraged to use animation rather than dragging the marker directly, because an object being animated can be slowed down so that it moves less than a pixel at a time.

Q3 The domain of function $v(a)$ is $a \geq 0$. (When the input marker is to the left of 0, the output marker disappears.)

The range is $v(a) \geq 0$.

Q4 Negative numbers are not in the domain because the square root of any negative number is undefined over the set of real numbers.

Q5 The domain restriction is $b \neq 2$, and the range restriction is $w(b) \neq 0$. The value 2 is excluded from the domain because it would result in division by 0. The value 0 isn't part of the range because the result of the division $\frac{1}{x-2}$ cannot be 0.

DOMAIN AND RANGE ON CARTESIAN GRAPHS

On page 5, the graph's properties are set to plot it discretely rather than continuously. If the plot were done continuously, the discontinuities would be connected with segments. To change whether a graph is plotted continuously or discretely, select the object and choose **Edit | Properties | Plot.**

Q6 One way to determine domain by looking at a Cartesian graph of a function is to imagine a vertical line that sweeps from left to right. Any location where the line touches the graph is part of the domain. Thus, if a vertical line crossing the x-axis at $x = 3$ touches the graph somewhere, 3 is part of the domain of that function. Any location where the line doesn't touch the graph at all is not part of the domain.

Similarly, to determine the range from a Cartesian graph, imagine a horizontal line sweeping from bottom to top. Anywhere it touches some part of the graph is part of the range; anywhere it doesn't, isn't.

EXPLORE MORE

Q7 To make the range of f all even numbers, use $f(x) = 2 \cdot \text{round}(x)$.

To make its range all odd numbers, use $f(x) = 2 \cdot \text{round}(x) + 1$.

Q8 To make the domain and range of v all numbers less than or equal to 0, use $v(x) = -\sqrt{-x}$.

To make w's domain all real numbers except 0 and its range all real numbers except 2, use $w(x) = \frac{1}{x} + 2$.

WHOLE-CLASS PRESENTATION

Open **Domain Range Present.gsp** and use the pages of this sketch to stimulate a class discussion.

1. On page 1, drag input marker a and ask students to observe the possible positions of the input and output markers, and use their observations to describe the domain and range. After several students have volunteered descriptions in their own words, turn on tracing and drag again to verify the descriptions. You can use the Animation button to achieve smooth movement of the input marker.
2. For each of the four functions on page 2, ask students to guess ahead of time what the range will be. Then drag the input marker for that function to test their guesses. For each function, use tracing and animation to generate a smooth, detailed visual representation of the answer.
3. Use page 3 to emphasize the need to pay attention to details and not to jump to conclusions. By dragging the input marker, you'll generate what appears to be a range of integer values only. Ask students to explain what's going on here. This should generate a lively discussion. Encourage a number of students to describe the phenomenon in their own words. Finish this page by using animation to achieve movement by less than a pixel at a time.
4. On page 4, tracing shows the domain and range restrictions for $v(a)$ clearly, but cannot show the restrictions for $w(b)$ so clearly. Get students to discuss the differences in the two situations, so that they realize that $w(b)$ is missing only a single number in its domain and a different number in its range.
5. On page 5, drag input marker a and have students observe both the dynagraph and Cartesian graph. Ask them to explain how the restricted range shows up on the Cartesian graph. Also ask them how to edit the function to generate only even numbers, or only odd numbers.
6. Use page 6 to compare the dynagraph and Cartesian representations for the functions from page 2.

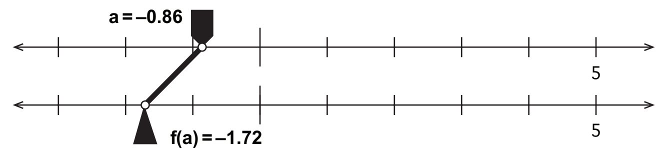
Finish by asking students to summarize what they learned about domain and range.

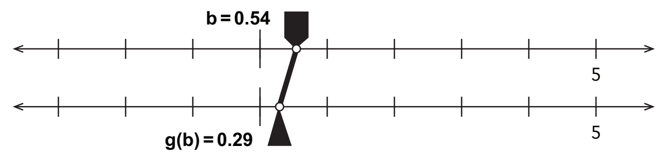
Function Composition with Dynagraphs

In life, the answer to one question sometimes becomes a question that leads to another answer. Functions are much the same; sometimes we take the output of one function and make it the input for a second function. This is called *function composition*, and we say that the two functions have been *composed*. In this activity you'll get a brief introduction to function composition and then see how dynagraphs can provide a compelling way of modeling composed functions.

INTRODUCTION

We'll introduce function composition informally by doing some examples with numbers.





The composite function $g(f(3))$ is pronounced "g of f of 3."

Given $f(x) = 2x$ and $g(x) = x^2$, find $g(f(3))$.

You always evaluate parentheses first, so start on the inside by evaluating $f(3)$: $f(3) = 2 \cdot (3) = 6$.

Take this output and make it g's input: $g(f(3)) = g(6) = (6)^2 = 36$.

That, in a tiny nutshell, is function composition.

Q1 Given the functions

$$f(x) = 2x,\ g(x) = x^2,\ h(x) = \text{round}(x), \text{ and } j(x) = \frac{x}{2},$$

evaluate the following expressions:

a. $g(f(5))$ b. $f(g(3))$ c. $f(h(3.6))$

d. $j(g(-6))$ e. $j(f(17))$ f. $f(j(17))$

Q2 Do you think $f(g(x))$ always equals $g(f(x))$? Answer this question by comparing $f(g(5))$ and $g(f(5))$.

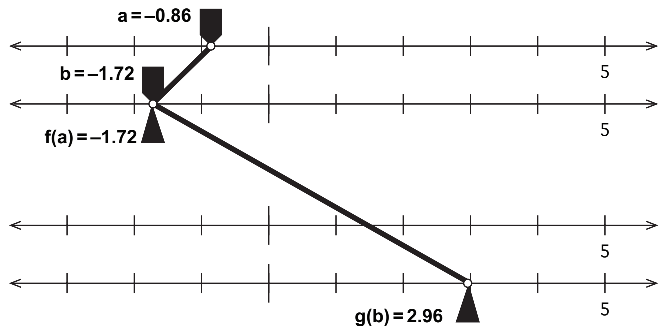
SKETCH AND INVESTIGATE

1. Open **Composite Functions.gsp.** Drag each input marker to familiarize yourself with the dynagraphs of functions f and g.

You'll now model $g(f(x))$. The trick is to use Sketchpad's **Split** and **Merge** commands so that the output of f becomes the input of g.

2. Select point b (at the tip of g's input marker) and choose **Edit | Split Point From Line.** The point is separated from its axis.
3. Select points b and $f(a)$. Choose **Edit | Merge Points.**

The two points are merged into one, and the output of f is the input of g. Drag the input marker of f to explore your new composite function, $g(f(x))$, and to check your answer from Q1 part a.



4. Go to page 2, which contains the same two functions, but with a g above and f below. Use the technique from steps 2 and 3 to model $f(g(x))$.

Q3 Use your composite dynagraph to evaluate these expressions:

a. $f(g(1))$ b. $f(g(-1))$ c. $f(g(-7))$

Next, you'll compose the "round" and "square" functions to create a composite function with an interesting set of outputs.

Q4 Go to page "h&g" and model $g(h(x))$. What is the range of this composite function? In other words, what are its possible outputs?

Use the remaining pages of the document to experiment with other composite functions.

Q5 Create the composite functions $j(f(x))$ and $f(j(x))$ on the appropriate pages. Experiment with these functions. What special feature do you notice about these two composite functions? Why does this happen?

EXPLORE MORE

1. There's nothing stopping you from composing more than two functions to get something such as $h(g(f(x)))$. Go to page "all 4" and try this for different combinations of three or four functions. See if you can build the following functions:

   a. a function that outputs twice perfect squares (0, 2, 8, 18, . . .)

   b. a function that outputs squares of even numbers (0, 4, 16, 36, . . .)

   c. a function that outputs the perfect squares divided by 4 (0, 0.25, 1, 2.25, 4, 6.25, . . .)

Objective: Students use dynagraphs to model composite functions, using the output of one function as the input to a second function.

Student Audience: Algebra 2/Precalculus

Prerequisites: It's helpful if students have already been introduced to dynagraphs by doing the first dynagraph activity: Introducing Dynagraphs. They should also be familiar with function notation.

Sketchpad Level: Easy. Students manipulate a pre-made sketch and add on to it minimally.

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity (use **Composite Functions.gsp**) or Whole-Class Presentation (use **Composite Functions Present.gsp**)

Related Activities: Introducing Dynagraphs, From Dynagraphs to Cartesian Graphs, Domain and Range

The term *dynagraph* was coined by Paul Goldenberg, Philip Lewis, and James O'Keefe in their study "Dynamic Representation and the Development of a Process Understanding of Functions" published by Education Development Center, Inc., and supported in part by a grant from the National Science Foundation. See the Activity Notes from the activity Introducing Dynagraphs for a more thorough discussion of dynagraphs in general.

Composite functions often seem backward to students. Since they read $g(f(x))$ left to right, they may be surprised to see that its composite dynagraph starts at the input marker of f and ends at the output marker of g. The reason, of course, is that when evaluating $g(f(x))$, you start with the parentheses and apply the functions from right to left, even though you read the expression from left to right. Make sure students really do understand why composite dynagraphs are structured as they are.

A possible discussion topic with students is the fact that function composition lurks behind *every* function. For example, the simple function
$f(x) = 2x + 1$ can be seen as $g(h(x))$, where
$g(x) = x + 1$ and $h(x) = 2x$, or
$g(x) = 2x$ and $h(x) = x + 0.5$, or
$g(x) = 2x + 1$ and $h(x) = x$, or
many other possibilities. Students may enjoy decomposing functions in this way, and the practice will be valuable when they learn to transform functions.

INTRODUCTION

Q1 a. 100 b. 18 c. 8
d. 18 e. 17 f. 17

Q2 No, $f(g(x))$ doesn't always equal $g(f(x))$. In this case, $f(g(5)) = 50$ and $g(f(5)) = 100$.

SKETCH AND INVESTIGATE

4. The only way to undo merging an input point with an output point is literally to undo. That is, choose **Edit | Undo** repeatedly (or hold down the Shift key and choose **Edit | Undo All**). Splitting (by choosing **Edit | Split**) won't work because the original output points were constructed in a way that does not allow them to be split.

Q3 a. 2 b. 2 c. 98

Q4 The range of $g(h(x))$ is all squares of whole numbers, or, (0, 1, 4, 9, 16, 25, 36, . . .).

A nice way to see this is to turn on tracing for the output marker (by selecting it and choosing **Display | Trace Triangle**) and then drag the input marker. This is, in fact, the method used in the activity Domain and Range.

Q5 The output is always the same as the input in both cases. This is because $f(x)$ and $j(x)$ are inverse functions of each other, meaning that they "undo" each other. In other words, start with any number, double it, then halve it, and you'll end up where you started. It works the same if you halve it first, then double it.

EXPLORE MORE

1. a. $f(g(h(x)))$
   b. $g(f(h(x)))$
   c. $g(j(h(x)))$

By representing composite functions on a dynagraph, it is possible to show a number graphically as it leaves one function and becomes the argument of a second function. Seeing the concept in another way may help students to understand the process as opposed to memorizing a rule.

1. Open **Composite Functions Present.gsp.** The first page shows two functions: $f(x) = 2x$ and $g(x) = x^2$. On each function, drag the input marker (the pentagon above the axis) and tell the class to observe the output.

Q1 What is the range of each function? (The range of f is all real numbers. The range of g is all non-negative real numbers.)

Explain that function composition means using the output from f as the input to g.

Q2 Drag the input marker for g so its value is reasonably close to the output for f. Ask if this is close enough. Get students to agree that you need a better method.

2. Select point b, the input of g. Choose **Edit | Split Point From Line.**
3. Select points b and $f(a)$. Choose **Edit | Merge Points.**

Q3 What we have done graphically is to set b equal to $f(a)$. Write "$b = f(a)$" in the sketch or on the board. Point to the marker for $g(b)$ on the sketch. Using substitution, what would be a logical name for this value? (Pronounce it slowly as you write "g of f of a.")

$$g(b) = g(f(a))$$

4. The next page is labeled "g&f". Use the same procedure to construct $f(g(b))$. Flip between the two pages as you compare the composite functions.

Q4 What are the ranges of $g(f(x))$ and $f(g(x))$? (All non-negative real numbers in both cases.)

Q5 Is $g(f(x)) = f(g(x))$ for all x? This is true only for $x = 0$, but challenge students to work it out themselves before showing a counter-example.

5. Go to the page labeled "f&j", which has the functions $f(x) = 2x$ and $j(x) = \frac{x}{2}$.

Q6 What are $f(j(x))$ and $j(f(x))$? (They are both the identity x.)

6. Perform the constructions to answer Q5.

You can undo all of the modifications to any page in the document. While holding down the Shift key, choose **Edit | Undo All.**

In this document, there are a total of four functions. They all appear on the last page, "all 4". Try the various combinations. In each case, ask students about the range of the composite function, and ask them whether the order of the functions matters.

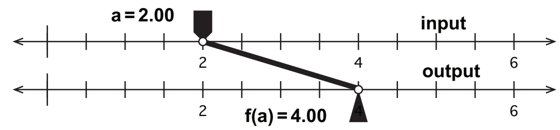
You can also customize these functions. Double-click on a function definition on the left side of the screen, and enter a new definition.

Odd and Even Functions

Just as there are odd and even numbers, there are odd and even functions. Unlike numbers, there are also functions that are neither odd nor even. Whether a function is odd, even, or neither depends on its symmetry. In this activity you'll explore odd and even functions using both dynagraphs and Cartesian graphs.

SKETCH AND INVESTIGATE

1. Open **Odd Even Functions.gsp.**



Q1 Use the first dynagraph to determine these values:

a. $f(2)$ and $f(-2)$ b. $f(-1)$ and $f(1)$

c. $f(3)$ and $f(-3)$ d. $f(-0.5)$ and $f(0.5)$

Q2 In each case, what do you notice about how $f(a)$ compares with $f(-a)$?

Functions like this, where the output for $-x$ is the *opposite* of the output for x, are said to be *odd* or to have *odd symmetry*. We can say the following:

An *odd function* is one in which $f(-x) = -f(x)$ for all x in its domain.

Q3 Of the other three functions modeled on page 1, which are odd?

Q4 On page 2, use the first dynagraph to determine these values:

a. $t(2)$ and $t(-2)$ b. $t(-1)$ and $t(1)$

c. $t(3)$ and $t(-3)$ d. $t(-0.5)$ and $t(0.5)$

Q5 In each case, what do you notice about how $t(x)$ compares with $t(-x)$?

Functions like this, where the output for $-x$ is the *same* as the output for x, are said to be *even* or to have *even symmetry*. We can say the following:

An *even function* is one in which $f(-x) = f(x)$ for all x in its domain.

Q6 Of the other three functions modeled on page 2, which are even?

Q7 Describe how to tell whether a dynagraph you're exploring represents an odd or an even function just by dragging (in other words, without looking at its equation).

Q8 Go to page 3. Model the three odd functions from page 1 and the three even functions from page 2 on the combination dynagraph/Cartesian graph. What do odd functions look like when plotted in the xy plane? How about even functions?

To enter round(x), $|x|$, sin(x) or cos(x) in the Edit Function dialog box, use the Functions pop-up menu. Don't worry if you're not familiar with the sine or cosine functions—you can tell whether they're odd or even just as with the other functions.

Q9 Model the following functions on the combination dynagraph/Cartesian graph. Some of the functions are odd, some are even, and one is neither. Use what you learned in Q8 to determine which are which, without using the dynagraph.

a. $f(x) = 5x$

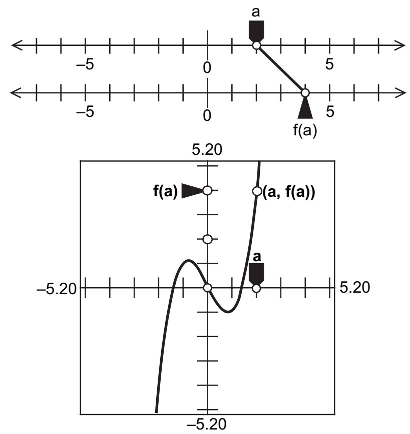
b. $f(x) = x^3 - 2x$

c. $f(x) = \sin(x)$

d. $f(x) = x^2 + 2$

e. $f(x) = x^4 - 3x^3$

f. $f(x) = \cos(x)$



EXPLORE MORE

On page 4, you will construct transformed images of a graph and compare odd and even functions with the transformed images of their graphs.

To reflect P, select the x-axis and choose **Transform | Mark Mirror.** Then select P and choose **Transform | Reflect.**

To construct the locus, select both P and its reflected image, and choose **Construct | Locus.**

2. Page 4 contains the graph of a polynomial function and a point P on the graph. Reflect P across the y-axis. Then construct the locus of the reflected point as P moves along the graph. Make the locus thick and light blue.

3. Construct another transformed image of the original graph, this time reflected across the x-axis. Construct a third transformed image of the original graph, this time reflecting P across first the y-axis and then the x-axis. Make each locus thick and a different light color from the others. Record your observations.

Q10 Edit the original function to match each of the functions listed in Q9. For each function, record your observations about the original function and the images.

Q11 Why are odd functions called "odd" and even functions called "even"? You may be able to figure this out by looking at the equations of the odd and even functions you've seen in this activity. (*Hint:* Focus on just the polynomial equations.) Test your answer by modeling your own odd and even functions in Sketchpad and verifying that they exhibit the characteristic symmetry.

Q12 Imagine that both $f(x)$ and $g(x)$ are odd functions. What kind of function will $(f + g)(x)$ be? What about $(f \cdot g)(x)$? What about $g(f(x))$? What will the results be if f and g are both even? What if one is even and the other odd?

Odd and Even Functions

Objective: Students explore odd and even functions using dynagraphs, identify characteristics of these functions in the Cartesian plane, and compare the original functions to transformed images of their graphs.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students should already be familiar with dynagraphs by having completed the activities Introducing Dynagraphs and From Dynagraphs to Cartesian Graphs.

Sketchpad Level: Easy. Students manipulate a pre-made sketch by dragging markers and editing functions.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Odd Even Functions.gsp**) or Whole-Class Presentation (use **Odd Even Functions Present.gsp**)

Before introducing odd and even functions, tell students that mathematicians often classify mathematical objects based on properties that make them unusual or special. For example, even numbers are special because they're always divisible by 2; prime numbers are special because they have no proper divisors. What properties might make a function special? Students have already seen some: The round(x) function is unusual because it contains breaks—you can't draw it without lifting your pencil. Such functions are called *discontinuous*. This discussion will set the stage for the investigation of odd and even functions.

One important thing when doing this activity is the difference between "negative" and "opposite." In the equation $f(-x) = -f(x)$ that characterizes odd functions, the input on the left side of the equation isn't necessarily negative—it's just the opposite of the input on the right side. Similarly, the output on the right side isn't necessarily negative—it's just the opposite of the output on the left side. It's important that students understand this point.

In Explore More steps 2 and 3 and Q10, students actually construct the reflected functions. This helps to make their conclusions about the symmetry of the graphs more concrete and more obvious to them. If your students are already comfortable with Sketchpad constructions, you should make this section of the activity mandatory for them. If not, consider using the presentation sketch to show these transformed images and explore their relationship with even and odd functions.

SKETCH AND INVESTIGATE

Q1 a. $f(2) = 4; f(-2) = -4$

b. $f(-1) = 1; f(1) = -1$

c. $f(3) = 21; f(-3) = -21$

d. $f(-0.5) = 0.88; f(0.5) = -0.88$

Q2 $f(a)$ and $f(-a)$ are opposites.

Q3 The function $j(x)$ is also odd. The function $h(x)$ is neither odd nor even.

Q4 a. $t(2) = 2; t(-2) = 2$

b. $t(-1) = -1; t(1) = -1$

c. $t(3) = 7; t(-3) = 7$

d. $t(-0.5) = -1.75; t(0.5) = -1.75$

Q5 $t(a)$ and $t(-a)$ are the same. For example, $t(3) = 7$ and $t(-3) = -7$.

Q6 The functions $u(x)$ and $v(x)$ are also even. The function $w(x)$ is neither even nor odd.

Q7 If it's odd, its behavior on the right side of the origin mirrors its behavior on the left. If it's even, an input to the right of the origin gives the same output as an input the same distance to the left of the origin.

Q8 Graphs of odd functions are symmetrical about the origin. If the graph is rotated by 180° around the origin, it will look exactly the same. (This symmetry is often called *point reflection.*)

Even functions are symmetrical across the y-axis, meaning that reflecting the graph across the y-axis will result in it looking exactly the same.

Q9 $f(x) = 5x$, $f(x) = x^3 - 2x$, and $f(x) = \sin(x)$ are all odd.

$f(x) = x^2 + 2$ and $f(x) = \cos(x)$ are both even.

$f(x) = x^4 - 3x^3$ is neither even nor odd.

EXPLORE MORE

2.–3. You cannot reflect a graph directly, but steps 2 and 3 provide an easy way to accomplish the same result, by transforming a single point and then constructing the locus of the transformed image. By making the transformed graph thick and light in color, the original dashed graph will remain clearly visible when the function is edited so that the two graphs coincide.

Q10 The results are the same as in Q9 but may be more obvious to students because the transformed image actually appears on the screen.

Q11 An easy way to tell if a polynomial is odd or even is to look at the degrees of its terms (meaning the values of the powers of x). If they're all odd, it's an odd function. This is why odd functions are called "odd." If the degrees are all even, it's an even function. This is why even functions are called "even." Constants (such as the 3 in $f(x) = x^2 + 3$) are considered even-degreed terms since they can be thought of as being a constant times x^0 (for example, $3 = 3x^0$).

There's no such easy way of classifying functions that are not polynomials (such as sin and cos functions).

Q12 If $f(x)$ and $g(x)$ are both odd, $(f + g)(x)$ is also odd. One way to see this would be to redefine $h(x)$ on page 1 as $f(x) + g(x)$ and then explore the resulting dynagraph. Another way is as follows: $(f + g)(-x) = f(-x) + g(-x) = -f(x) + -g(x) = -[f(x) + g(x)] = -(f + g)(x)$. Look at the first and last expressions to see that $(f + g)(x)$ is indeed odd. Using one of these methods, you can see that if $f(x)$ and $g(x)$ are both odd, $(f \cdot g)(x)$ must be even.

The composition $g(f(x))$ is even if f is even, because a and $-a$ both produce the same output from f. Because the output of f is the input to g, the output of g will also be the same for both a and $-a$. If f is odd, the output of f (and the input to g) is opposite for a and $-a$. The result of the composition is odd if g is odd, even if g is even, and neither if g is neither.

If f is neither even nor odd, there is no way to draw any general conclusions about the composition.

WHOLE-CLASS PRESENTATION

Use this presentation to get students to think about characteristics of functions and to view certain types of symmetry that functions can exhibit.

Q1 Open **Odd Even Functions Present.gsp** and ask students to observe the output of the first function (labeled *Even*) as you change the input to its opposite. Press the $a => -a$ button and ask students what they observe. Press the button again to let them watch again while a returns to its original position.

Q2 Ask students to make a conjecture about the behavior of even functions.

Q3 To test the conjecture for many values of a, press the *Show f(–a)* button and drag input a. Ask students whether the output behavior supports the conjecture.

Q4 Similarly, use the $b => -b$ button and ask students to make a conjecture about the behavior of odd functions. Test the conjecture for many values of c.

Q5 Use the $c => -c$ button and ask students what pattern they observe in the output. Test for many values of b.

Q6 On page 2, use dragging and/or the buttons to show the behavior of these functions one at a time. Ask students to determine whether each function is odd and to explain their conclusion. Similarly, have students determine which of the functions on page 3 are even.

Q7 On page 4, drag the input marker and ask students to characterize the function $2x - 1$. (It's neither even nor odd.) Edit the function to be $f(x) = 5x$, and ask students to predict whether the function will be even, odd, or neither. Drag the input marker, and then press the *Show f(–a)* button and drag again. Ask students to describe in detail what they observed, and ask them to categorize this function as even, odd, or neither. Repeat for the functions below:

a. $f(x) = 5x$
b. $f(x) = x^3 - 2x$
c. $f(x) = \sin(x)$
d. $f(x) = x^2 + 2$
e. $f(x) = x^4 - 3x^3$
f. $f(x) = \cos(x)$

Q8 Ask students what they notice about the four polynomial functions they just categorized. Make a list of which ones were even (d), which were odd (a and b), and which were neither (e). Ask them if they have any ideas about why the names *even* and *odd* are used for the behaviors they observed.

Q9 On page 5, ask students whether they think the function shown is even, odd, or neither. Then ask them which reflection will coincide with the original graph—a reflection across the y-axis or one through the origin. Show each reflection in turn, to check students' answers. Then reset the reflections, edit the function, and repeat.

Finish with a class discussion connecting the behavior of the dynagraphs, the shape of the Cartesian graphs, and the reflections of those graphs.

Inverse Functions

Most arithmetic operations have inverses. If you add 3 to one number to get a second number, you can get back to your original number by adding -3 to your new number. Similarly, if you multiply one number by 2 to get a second number, you can get back to your original number by multiplying by 1/2. We say that -3 is the *additive inverse* of 3 and that 1/2 is the *multiplicative inverse* of 2.

Similarly, many functions have inverses, allowing you to start with the output of a function and get back the input. In this activity you'll explore the inverses of several functions and figure out when inverse functions exist and when they don't.

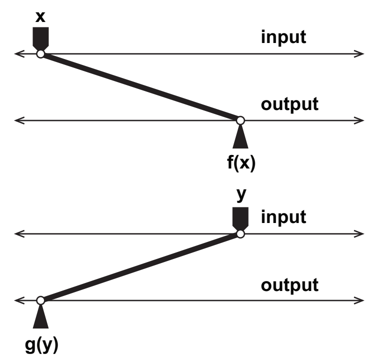
INVESTIGATE

1. Open **Inverse Functions.gsp.** You'll see two dynagraphs.

There are no numbers on the axes, but you can observe the value of *x* to move the input marker where you want it.

Q1 Drag the $f(x)$ input marker to 1.50. What is the output of $f(x)$?

Q2 Drag the $g(y)$ input marker to the same value as the $f(x)$ output. How does the output of $g(y)$ compare to the input of $f(x)$?



To redefine the function, double-click it and enter a new function definition in the Calculator.

Q3 How can you redefine $g(y)$ so its output is the same as the $f(x)$ input? Do it now.

2. Select the input and output values of $f(x)$ and the input and output values of $g(y)$. Then choose **Graph | Tabulate.** Double-click the table to make the current values permanent.

3. Drag the $f(x)$ input marker to a different value. Then drag the $g(y)$ input marker to match the $f(x)$ output.

Q4 How does the output of $g(y)$ compare to the input of $f(x)$? Write down the input and output of both functions. Double-click the table to record these values.

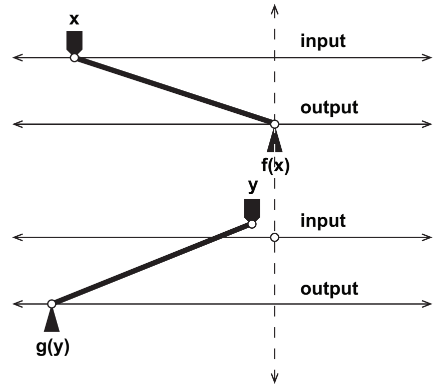
4. Try three new values of the $f(x)$ input. Each time set the $g(y)$ input to match the $f(x)$ output, and record your values in the table.

Q5 What can you conclude about these two functions?

You can test your conjecture for many input values by attaching the $g(y)$ input to the $f(x)$ output. This is called the *composite function,* written as $g(f(x))$.

5. In this step you'll mark the $f(x)$ output on the $g(y)$ input axis. First drag the $f(x)$ input to a new value. Then select both the $f(x)$ output axis and the $f(x)$ output point (at the top of the output marker), and choose **Construct | Perpendicular Line.** Construct a point at the line's intersection with the $g(y)$ input axis by clicking the **Arrow** tool on the intersection.

6. Attach the $g(y)$ input to this point by selecting the $g(y)$ input point (at the bottom of the input marker) and choosing **Edit | Split Point From Line.** Then select both the intersection and the $g(y)$ input point, and select **Edit | Merge Points.**



7. Drag the $f(x)$ input to make sure that the attachment worked. Record four new sets of values in your table.

The inverse of $f(x)$ is often written as $f^{-1}(x)$.

If the output of $g(y)$ always matches the input of $f(x)$, f and g are inverse functions.

To change the definition of a function, double-click the function and enter the new definition.

8. Change the definition of $f(x)$ to $x - 1$.

Q6 Are the functions still inverses? If not, how could you change the definition of $g(y)$ to make them inverses again? Change the definition of $g(y)$ to check your answer.

Q7 Change the definition of $f(x)$ to $2x$. What operation do you think you should use so that the output of $g(y)$ matches the input of $f(x)$?

9. Change the definition of $g(y)$, and check your result by trying many values of x.

Q8 What is the inverse of $f(x) = 2x$? Write your answer using f^{-1} notation.

Q9 Change $f(x)$ to $\frac{2}{x}$ and figure out a definition of g that makes f and g inverses. Can you find any values of x for which the input of f doesn't match the output of g? Explain your results.

Q10 Find the inverse of each of the following functions. Check your answers using the dynagraphs, and describe any limitations that you find.

a. $f(x) = 2x - 1$

b. $g(x) = \frac{x}{3} + 1$

c. $h(x) = 3(x + 2)$

d. $p(x) = \frac{(x - 2)}{4}$

e. $q(x) = x^2$

f. $r(x) = x^2 - 2$

Q11 What unusual features do you notice about the last two inverses from Q10?

EXPLORE MORE

Q12 When f and g are inverse functions, what can you say about the behavior of the composite function $g(f(x))$? Use page 2 of the sketch to investigate.

Q13 If g is the inverse of f (written as $g = f^{-1}$), what does that mean about g^{-1} (the inverse of g)? To find out, go to page 3 and follow the directions there.

Objective: Students use linked dynagraphs to investigate inverse functions. They define inverses for various functions and test their results by inspecting values in a table and by observing the geometric behavior of the dynagraphs.

Student Audience: Algebra 1/Algebra 2

Prerequisites: None

Sketchpad Level: Easy. Students manipulate a pre-made sketch.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (use **Inverse Functions.gsp**) or Whole-Class Presentation (use **Inverse Functions Present.gsp**)

NOTATION

Point out that the variable x in $f(x) = x + 3$ is a placeholder and that $f(y) = y + 3$ represents exactly the same function. Page 1 of the sketch uses $f(x)$ and $g(y)$ to reduce confusion, but the other two pages use $f(x)$ and $g(x)$.

Point out that f is often used as a shorthand for $f(x)$. This shorthand appears in several places in the activity.

Discuss with students the use of $f^{-1}(x)$ to indicate the inverse of $f(x)$. This notation is used in several places in the activity. It's not used throughout because the sketch contains two different function definitions, which may or may not be inverses at any particular time. There's no guarantee that the functions are inverses, so they are identified as f and g rather than as f and f^{-1}.

INVESTIGATE

Q1 When $x = 1.50$, $f(x) = 4.50$.

Q2 When $x = 4.50$, $g(y) = 3.50$.

Q3 To make $g(y) = x$, redefine $g(y) = y - 3$.

Q4 The output of $g(y)$ is equal to the input of $f(x)$. Specific numeric examples will vary.

Q5 The function of $g(y)$ allows you to start with the output of $f(x)$ and get back the input. In other words, $g(f(x)) = x$. Encourage students to refer to their table data in thinking about this question.

Q6 After changing the definition of $f(x)$, the functions are no longer inverses. Changing the definition of $g(y)$ to $x + 1$ makes them inverses again.

Q7 Because x is multiplied by 2, we will have to use division by 2 to form the inverse.

Q8 The inverse of $f(x) = 2x$ is $f^{-1}(x) = x/2$.

Q9 The inverse of $f(x) = 2/x$ is $f^{-1}(x) = 2/x$. In this example, the function is its own inverse. This does not work for the value $x = 0$, because $f(x)$ is undefined when $x = 0$.

Use Q10 and Q11 to introduce the idea that some inverse functions do this successfully for the entire domain of the original function, and others do it for a subset of the domain. If students are familiar with the vertical line test, encourage them to think about how this relates to inverses. (Functions that are invertible over their entire domain also pass a "horizontal line test.")

Q10 Here are the inverse functions:

a. $f(x) = 2x - 1$	$f^{-1}(x) = (x + 1)/2$
b. $g(x) = x/3 + 1$	$g^{-1}(x) = 3(x - 1)$
c. $h(x) = 3(x + 2)$	$h^{-1}(x) = x/3 - 2$
d. $p(x) = (x - 2)/4$	$p^{-1}(x) = 4x + 2$
e. $q(x) = x^2$	$q^{-1}(x) = \sqrt{x}$
f. $r(x) = x^2 - 2$	$r^{-1}(x) = \sqrt{x + 2}$

Q11 The last two functions are inverses only for the domain $x \geq 0$. Outside this domain, the inverses give the wrong result because they return the positive square root. There is no way the inverse function can know when it should use the positive square root and when it should use the negative square root.

EXPLORE MORE

Q12 When $f(x)$ and $g(y)$ are inverse functions, the composite function $g(f(x))$ is the identity function: $g(f(x)) = x$. In other words, it's a function for which the output is always equal to the input.

Q13 If $g(y)$ is the inverse of $f(x)$, then $f(x)$ is also the inverse of $g(y)$. If there's a limitation on the domain of one of the two functions, there's a limitation on the range of the other.

In this presentation you'll use dynagraphs to find and present the inverses of several functions, and to figure out when inverse functions exist and when they don't.

PRESENT

1. Open **Inverse Functions Present.gsp.** Show the directions and buttons. This sketch contains dynagraphs of two functions, $f(x) = x + 3$ and $g(y) = y - 3$.

For easier operation, you can press the buttons provided on the sketch rather than dragging the input markers by hand.

Q1 Drag the top input marker (x) to change the input variable. Ask students, "For this value of x, what's the output of $f(x)$?"

Q2 Drag the bottom input marker (y) to match the output of $f(x)$. Ask students, "For this value of x, what's the output of $f(x)$?"

2. Select the input and output values of both functions, and choose **Graph | Tabulate.** Double-click the table to make the current values permanent.

3. Drag the x input marker to a different value. Then drag the y input marker to match the $f(x)$ output.

Q3 Ask students how the output of $g(y)$ compares to the input value x. Double-click the table to record these values.

4. Try several new values of x. Each time set y to match the $f(x)$ output and record your values in the table.

Q4 Ask students to make a conjecture concerning these two functions.

You can test your conjecture for many input values by attaching the y input to the $f(x)$ output. This is called the *composite function,* written as $g(f(x))$.

5. Use the *Attach y to f(x)* button, or the directions in steps 5–7 of the student activity sheets, to attach the input of g to the output of f. Drag x again and record four new sets of values.

Q5 Ask students to explain whether the table data confirm their conjecture.

Use page 2 of the sketch to define and examine inverses for several other functions. Pay particular attention to $f(x) = x^2$, and have students describe the problems or limitations they observe in their own words.

Use page 3 to investigate another problem with a surprising feature.

Use page 4 to look at the connection between inverses and the identity function.

Use page 5 to determine the inverse of f^{-1}.

Functions Again and Again

Imagine you have a long piece of licorice you want to split into eight equal pieces to share with seven friends. First, you break the licorice in half. Then, you break the halves in half. And finally, you break the smaller pieces in half for a total of eight equal-sized pieces.

The word for repeating a process over and over again is *iteration*. In the situation described above, you took an object, the licorice, and *iterated* an operation on that object, breaking the piece(s) in half. In this activity you'll iterate arithmetic operations on the coordinates of points.

SKETCH AND INVESTIGATE

To adjust the scale, drag a number on either axis.

After measuring, the point should be labeled *A*. If not, use the **Text** tool to change its label to *A*.

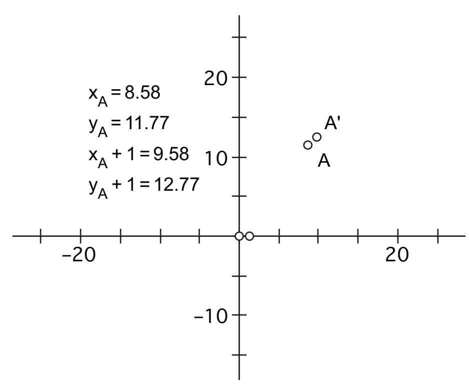
1. In a new sketch, choose **Graph | Define Coordinate System.** Adjust the scale until the x-axis goes from about -50 to 50 units.
2. Use the **Point** tool to construct a point anywhere in the plane. Measure the x- and y-coordinates of the point by choosing **Measure | Abscissa (x)** and **Measure | Ordinate (y).**

Q1 Without doing any calculations or drawings, imagine adding 1 to both the x- and y-coordinates and plotting the new point. Then imagine adding 1 to both coordinates of the new point to get another new point, and so forth. How would the points be arranged? If you connected them, what shape would they make?

You will use Sketchpad to check your prediction.

Choose **Measure | Calculate** to open the Calculator. Click the measurements in the sketch to enter them into the calculation.

3. Choose **Measure | Calculate** to calculate $x_A + 1$. Similarly, calculate $y_A + 1$. The results are the coordinates of your new point.
4. Use the **Arrow** tool to select the new x-coordinate ($x_A + 1$) and the new y-coordinate ($y_A + 1$) in order. Then choose **Graph | Plot As (x, y)** to plot your new point. Label the new point A'.

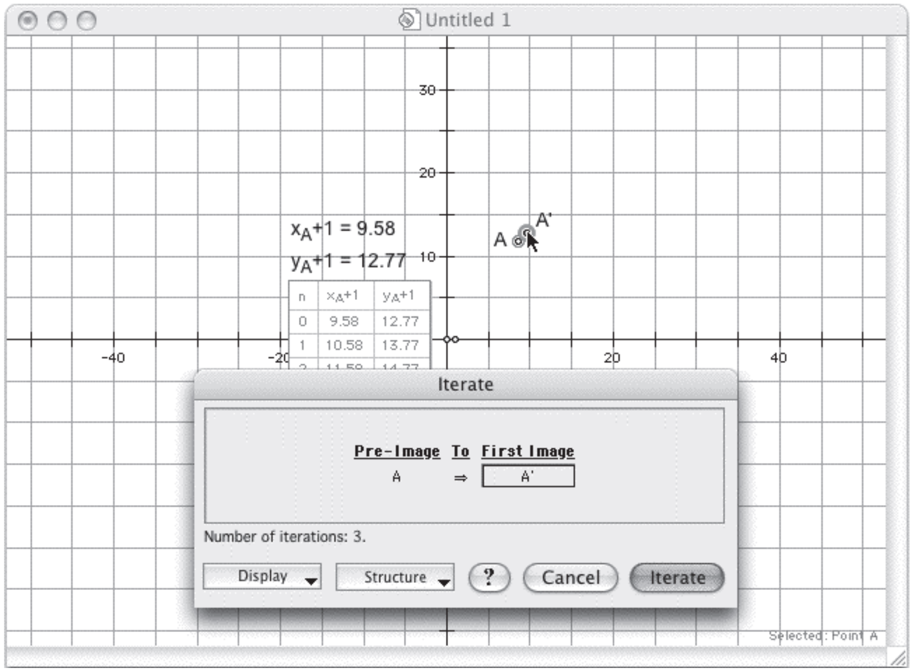


Select the two measurements and choose **Display | Hide Measurements.**

5. Hide the x_A and y_A coordinate measurements.

Now you will use iteration to create more points.

6. Select point A and choose **Transform | Iterate.** You'll get a dialog box that asks you to map point A to another object. Click point A' in the sketch, to indicate that the same operation should next be applied to A'. Click Iterate.



In the sketch itself, Sketchpad plots several more points by repeating (iterating) the operation that maps point A to point A'. Sketchpad also creates a table showing the calculations for each iterated point.

7. Select one of the iterated points, and press the + key several times. This will increase the number of iterations. Increase it until there are at least ten points.

The sequence is called the *orbit* of *A*.

Q2 The moment of truth has arrived! Drag your initial point A around. Describe the sequence of images generated by this iteration.

A function is a set of ordered pairs in which each first element (x) is associated with only a single second element (y).

Q3 As you drag point A, what happens to the values in the table? What changes and what stays the same? Do these values represent a function? Why or why not?

Q4 Imagine a line through your iterated sequence of points. What is the slope of this line?

Q5 Move point A as close as you can to (0, 2). What is the y-intercept of your imaginary line? Write an equation for the line. Check your result by choosing **Graph | Plot New Function** and plotting your equation. Once you have it right, delete or hide the new function you created.

Now change the iteration rule by changing the way y-coordinates are generated. Instead of adding 1 to the y-coordinate, what if you were to add 4 or -3?

To change a calculation, double-click it, and then change the expression in the Calculator.

Q6 Change the rule $(y_A + 1)$ to $(y_A + 4)$, and then to $(y_A - 3)$. How do these changes affect both the table and the slope of the imaginary line?

8. You can also change the iteration rule by changing the way you generate new *x*-coordinates. Try adding 2 to the *x*-coordinate. Try subtracting 0.5. Can you explain what happens?

Q7 What other combinations of *c* and *d* in the iteration rules $(x_A + c)$ and $(y_A + d)$ give the same slope you found in Q4?

Q8 Choose **Graph | Plot New Function.** Plot the equation $f(x) = 3x + 2$. Can you match your iteration to this plot by changing the starting position and the operations that generate new *x*- and *y*-coordinates? Describe how you did this.

Q9 Plot the equation $f(x) = -0.5x - 1$. How can you match your iteration to this plot? Delete or hide the function definitions when you are done.

EXPLORE MORE

Q10 How could you change the iteration rules so that the values in the table do not define a function?

So far you've generated only linear sequences of points, but there are many other types of sequences you can generate with this simple iteration rule. Some look quite strange, and some look like functions you're already familiar with.

If you want to increase the number of iterations for this investigation, select the iteration (by clicking on one of its points) and press the **+** key on your keyboard.

Q11 Instead of adding values to y_A, you could multiply it by a value. For example, instead of using $(y_A + 1)$ you could try $2 \cdot y_A$. (For now, keep adding 1 to the *x*-coordinate.) You can express this iteration rule more simply like this:

$$x' = x + 1, \quad y' = 2 \cdot y$$

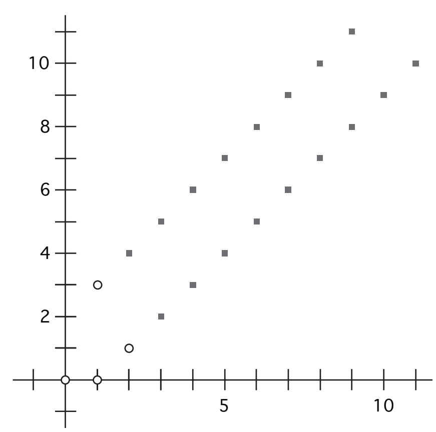
Drag *A* and describe the pattern of the iterated points. Then investigate the patterns generated by definitions in this form:

$$x' = x + 1, \quad y' = 2 \cdot y + k$$

Q12 Another interesting variation to try is to reverse the coordinates, using a rule like this:

$$x' = y, \quad y' = x + 1$$

Describe this sequence mathematically. Do the coordinates produce a function?



Functions Again and Again

Objective: Students define an iterated coordinate transformation on a point, observe the orbit, decide whether the iterated coordinates represent a function, and describe the function using a linear equation.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should be familiar with slope and have a basic understanding of functions.

Sketchpad Level: Intermediate. Students learn how to use Sketchpad's **Iterate** command.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Functions Again Present.gsp**)

SKETCH AND INVESTIGATE

In addition to the function that relates the *y*-coordinate of each point to the *x*-coordinate, there is another function involved in this activity: the function that transforms a point to its image. This function takes an input point (for instance, point *A*) and generates an output point (for instance, A'). Such functions can be used to create iterated function systems and can produce striking images like the Barnsley Fern. (See the Barnsley Fern activity in *Exploring Precalculus with The Geometer's Sketchpad.*)

2. After measuring the abscissa, students must deselect the new measurement, then reselect the point and measure its ordinate.

Q1 Answers will vary. It's important for students to try to imagine the results before doing the activity.

6. Students may need to drag the dialog box in order to see the point on which to click.

Q2 The sequence of iterated images is a set of points that lie on a ray starting from point *A*.

Q3 The values change according to the location of point *A*. Even though all the numbers change, the difference from one row to the next is always 1. The values do indicate a function, because each *x*-value corresponds to a single *y*-value.

Q4 The slope of the line is 1.

Q5 The *y*-intercept is 2, and the equation is $y = x + 2$.

Q6 When the rule is changed to $(y_A + 4)$, the slope is 4 and the *y*-values increase by 4 from one row to the next. When the rule changes to $(y_A - 3)$, the slope is -3 and the *y*-values decrease by 3 from one row to the next. When the rules are $(x_A + 1)$ and $(y_A + k)$, the slope is k.

Q7 The slope in Q4 is 1. The slope of the imaginary line produced by the rules $(x_A + c)$ and $(y_A + d)$ is d/c. Therefore, when $c = d$, the slope will equal 1.

Q8 Answers will vary. Students must change *c* and *d* so that $d/c = 3$, and must drag point *A* to be on the desired line. A logical position for *A* is at (0, 2).

Q9 Answers will vary. Students must make $d/c = -1/2$ and drag *A* appropriately—for instance, to $(0, -1)$.

EXPLORE MORE

Q10 If the rule for *x* is $(x_A + 0)$, all the *x*-values will be the same and will no longer define a function.

Q11 Repeated multiplication by 2 produces an exponential curve. Starting at (0, 1) produces 1, 2, 4, 8, 16, . . . corresponding to $y = 2^x$. A negative initial *y*-value results in a graph that curves downward rather than upward. You can generalize this principle using these definitions:

$$x' = x + 1, \quad y' = by, \quad \text{where } b > 0$$

The points fit the curve $y = ab^x$. The *y*-intercept is *a*.

The second definition in this section renders a more complicated pattern. Unlike a simple exponential curve, this sequence of points can cross the *x*-axis.

Q12 The new iteration forms two parallel rows of points. The lines containing the points have a reflection symmetry across the line $y = x + 1/2$. If you construct segment AA' before iterating, you can see the sequence and the symmetry more clearly. The coordinates define a function unless the fractional part of the initial *x*- and *y*-values are equal. If they are equal, there will be some *x*-values that correspond to more than one *y*-value.

WHOLE-CLASS PRESENTATION

Use the buttons and hints in **Functions Again Present.gsp** to present this activity to the class.