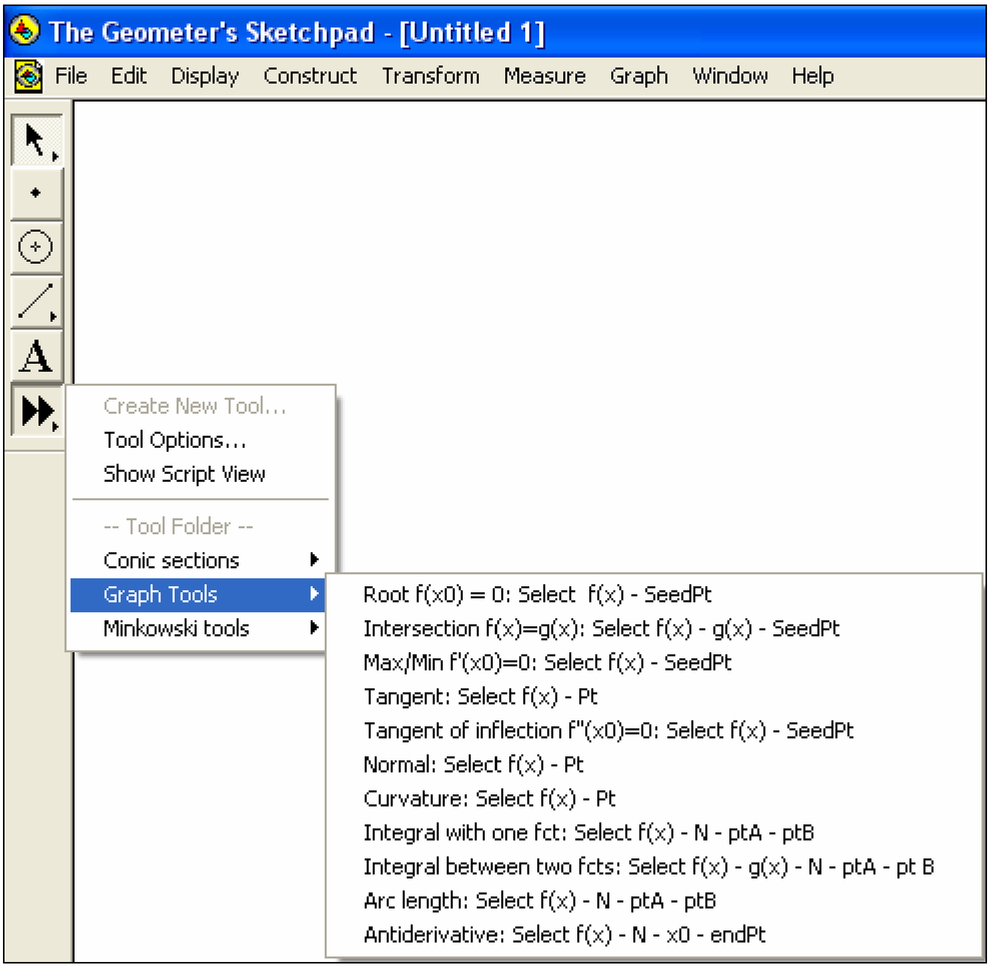
Graphing Calculator Tools for Sketchpad

Bjørn Felsager, Bjoern.Felsager@haslev-gym.dk, 2004

Since Sketchpad 4 is fully equipped with a graphing calculator – including the option of symbolic differentiation – it is now even possible to include a tool package for investigating function graphs with respect to roots, max/min, slopes, areas etc. Such a tool package will turn Sketchpad into a fully equipped graphing calculator allowing you to investigate numerically as well as dynamically most standard problems in elementary calculus. The tool package consists of an ordinary Sketchpad file with the extension **.gsp** having the name **Graph Tools**. You can download it from the official website of Sketchpad www.keypress.com/sketchpad. If you download the file **Grap Tools.gsp** into the folder **Tool Folder** these tools are automatically included when you open the program. But you can also open the file and run it in the background of a sketch, when you want to have access to the tools:



As you can see the package includes 11 standard tools which naturally fall into two groups: The first 7 tools deal with various applications of differentiation (typically invoking Newton's method). The last 4 tools deal with integration. Since Sketchpad are capable of performing differentiation symbolically it is fairly trivial to implement the first 7 tools. On the other hand no integration methods are supplied. Such numerical methods are less trivial to implement. They are partly explained in the sample documents about Riemann sums that are included with the program, and partly in an appendix devoted to the mathematics behind the Gauss summation method.

First part: Graph tools using differentiation

1. Root: Root f(x0) = 0: Select f(x) - SeedPt

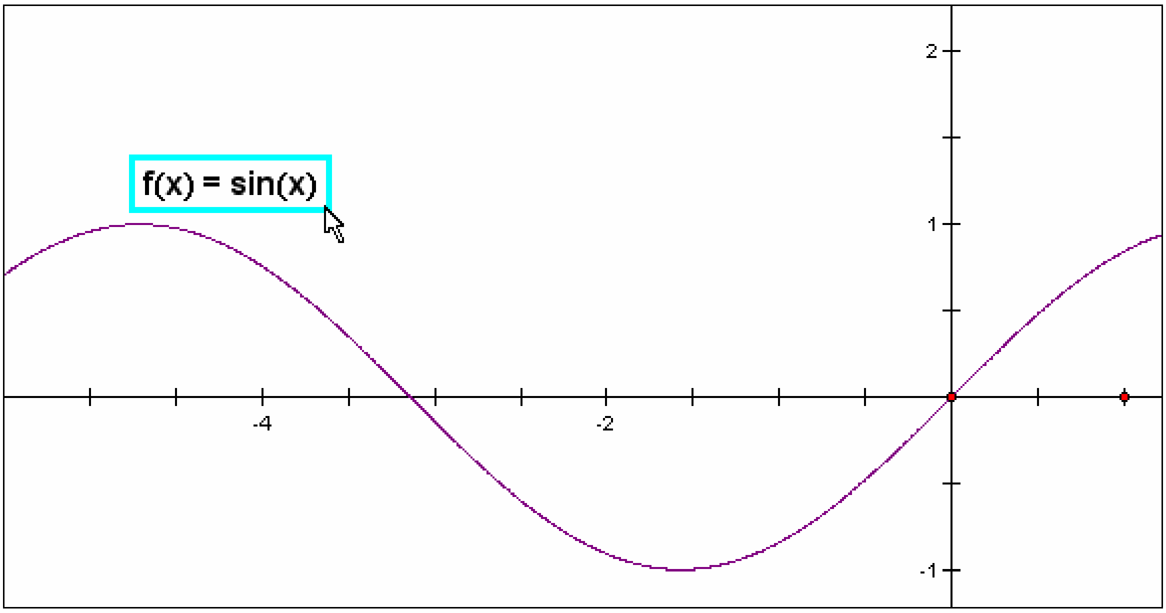
To determine a root we use Newton's method with 10 iterations

$$x \to x - \frac{f(x)}{f'(x)}$$

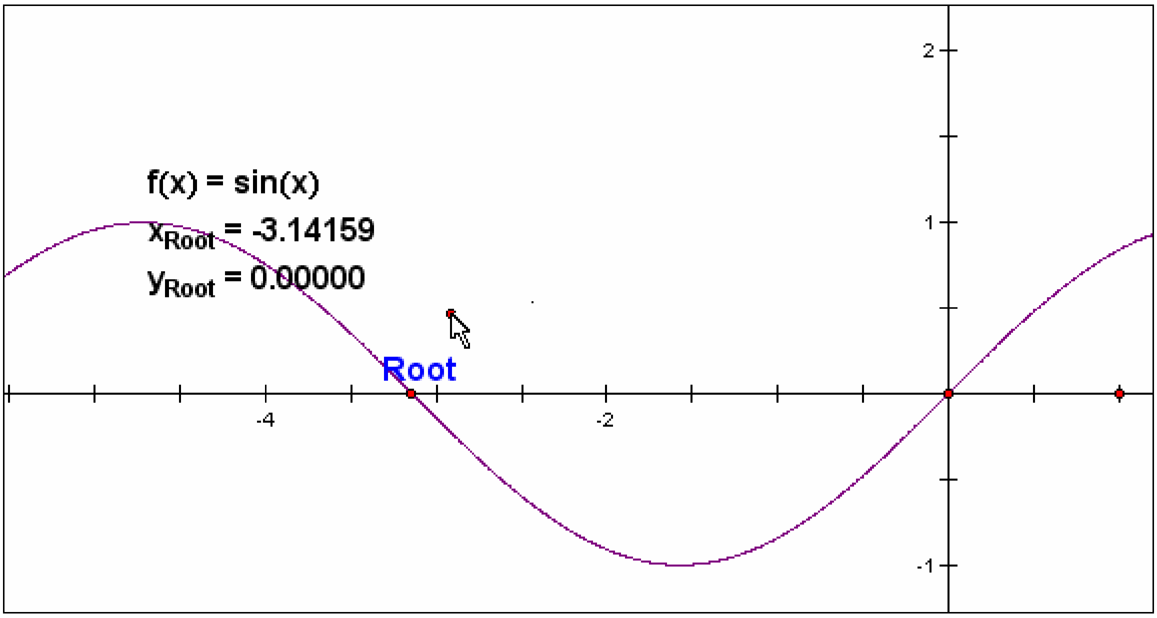
This presupposes a seed point fairly close to the desired root.

When selecting the **root tool** you must therefore first select the function $f(x)$ and then select a seed point close to the root. Once the function has been selected the root pops up close to the mouse pointer together with is coordinates.

1 step: Select the function:



2 step: Point out the root:



Remark: The y-coordinate is displayed as a check.

2. Intersection: Intersection f(x)=g(x): Select f(x) - g(x) - SeedPt

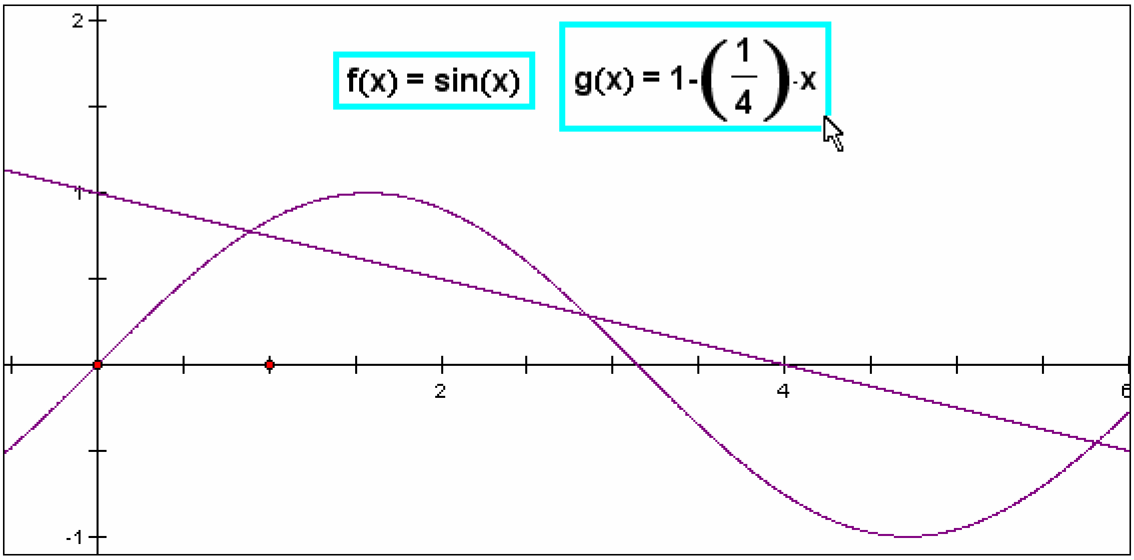
To determine the intersection points for two graphs we use Newton's method with 10 iterations to solve the equation $f(x) = g(x)$:

$$x \to x - \frac{f(x) - g(x)}{f'(x) - g'(x)}$$

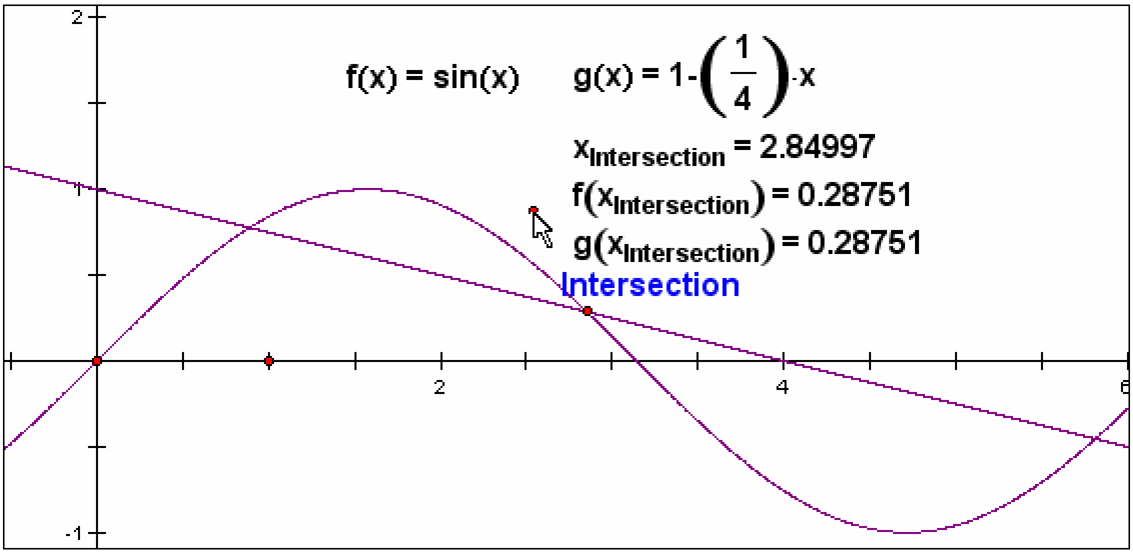
This presupposes that you select a seed point fairly close to the desired intersection point.

When selecting the **intersection tool** you must therefore first select the two functions $f(x)$ and $g(x)$ and then point out a seed point close to the intersection point. Once the functions have been selected the intersection point pops up close to the mouse pointer together with is x-coordinate as well as the values of $f(x)$ and $g(x)$ (as a numerical check of the equation solving).

1 step: Select the functions $f(x)$ and $g(x)$:



2 step: Point out the intersection point:



3. Max/Min: Max/Min f'(x0)=0: Select f(x) - SeedPt

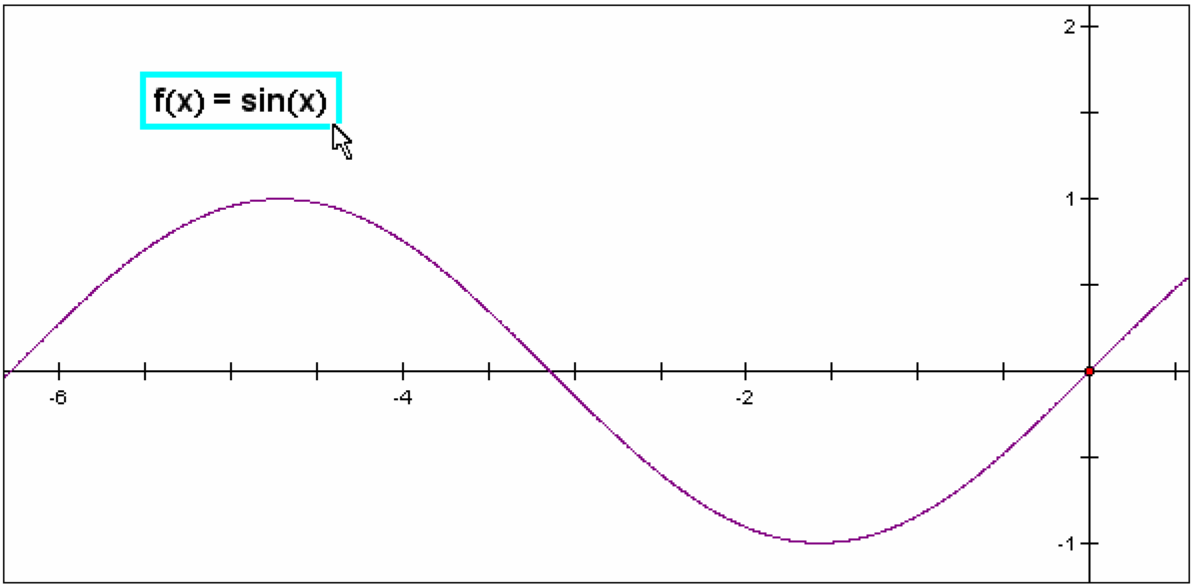
To determine stationary points we use Newton's method with 10 iterations to solve the equation $f'(x)=0$:

$$x \to x - \frac{f'(x)}{f''(x)}$$

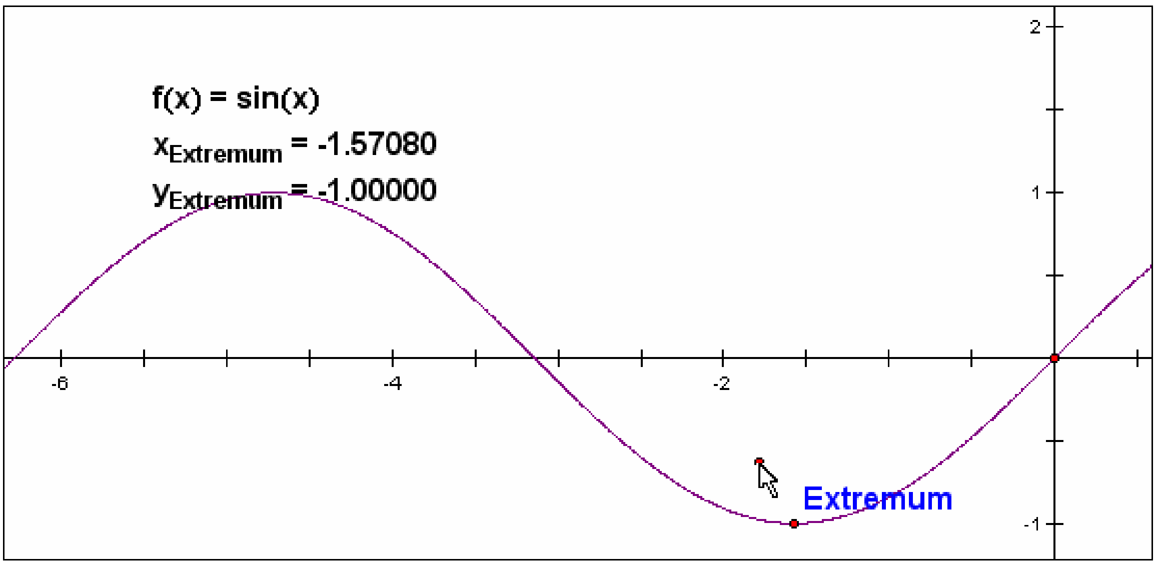
This presupposes a selection of a seed point fairly close to the desired stationary point.

When selecting the **max/min tool** you must therefore select the function $f(x)$ and then point out a seed point close to the stationary point. Once the function has been selected the stationary point pops up on the screen close to the mouse pointer together with its coordinates.

1 step: Select the function:



2 step: Point out the extremum:



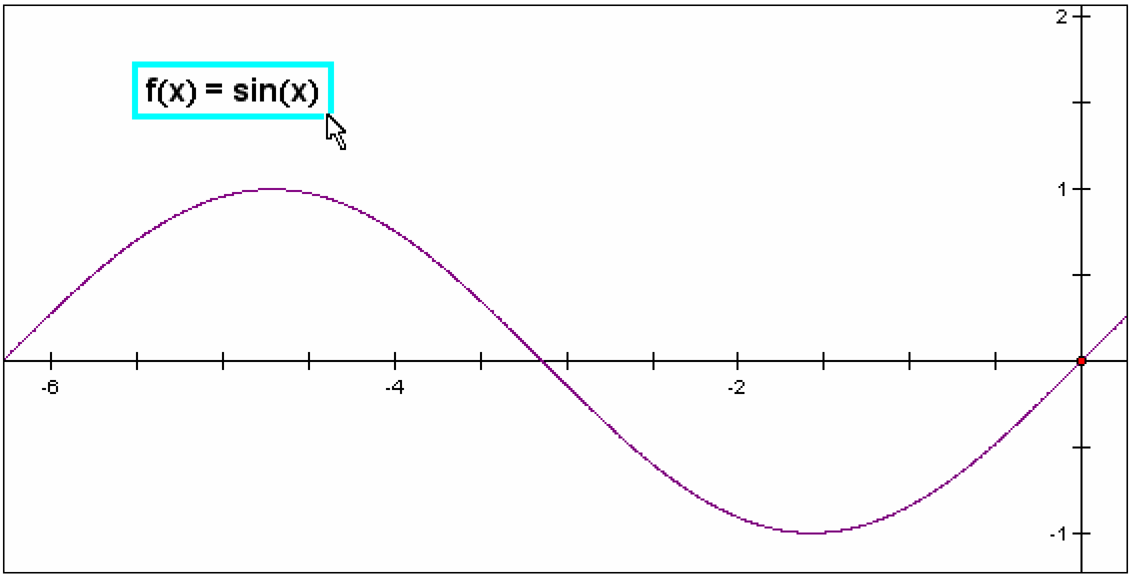
4. Tangent: Tangent: Select f(x) - Pt

To determine a tangent we use its slope $f'(x_0)$ as well as its point of contact $(x_0, f(x_0))$. This leads to the displaced point on the tangent:

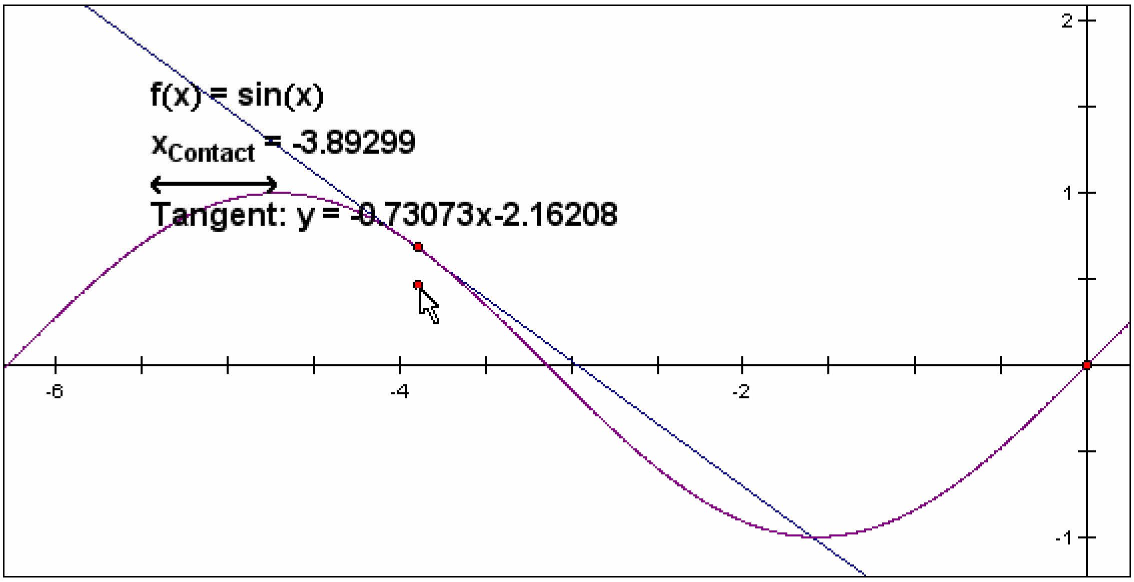
$$(x_0 + 1, f(x_0) + f'(x_0))$$

When selecting the **Tangent tool** you must therefore first select the function $f(x)$ and then point out the point of contact for the tangent. Once you have selected the function the tangent pops up on the screen at the graph point with the same *x*-coordinate as the mouse pointer together with the equation of the tangent as well as the *x*-coordinate for the point of contact.

1 step: Select the function:



2 step: Point out the point of contact for the tangent:



Remark: The above tangent has been constructed as a geometric line. If you need the tangent as a graph of a taylor polynomial (e.g. to be able to compute intersection points with the tangent) you should plot a new function entering the symbolic equation for the tangent instead: $f(x_0) + f'(x_0) \cdot (x - x_0)$.

5. Tangent of inflection: Tangent of inflection f''(x0)=0: Select f(x) - SeedPt

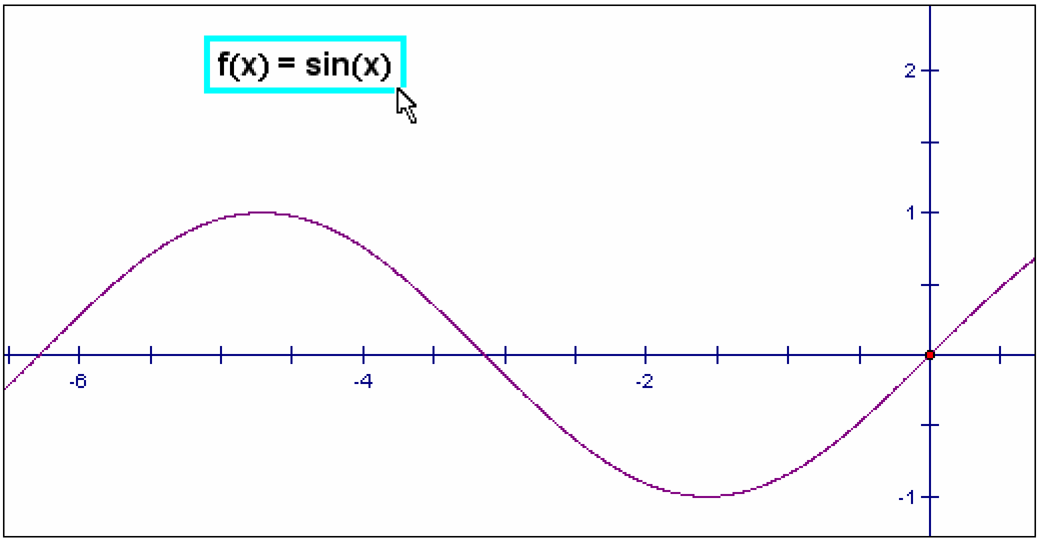
To determine a point of inflection together with its associated tangent we use Newton's method with 10 iterations to solve the equation $f''(x) = 0$:

$$x \to x - \frac{f''(x)}{f'''(x)}$$

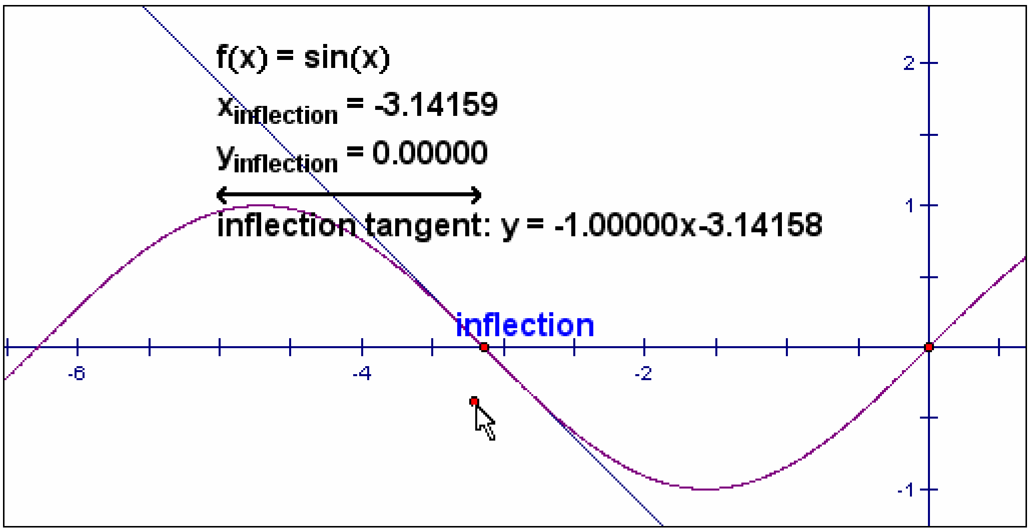
This presupposes that you select a seed point fairly close to the desired inflection point.

When selecting the **inflection tool** you must therefore select the function $f(x)$ and then point out the inflection point. Once you have selected the function the inflection point with its associated tangent pops up on the screen close to the mouse pointer together with its equation as well as the coordinates of the point of inflection.

1 step: Select the function:



2 step: Point out the point of inflection:



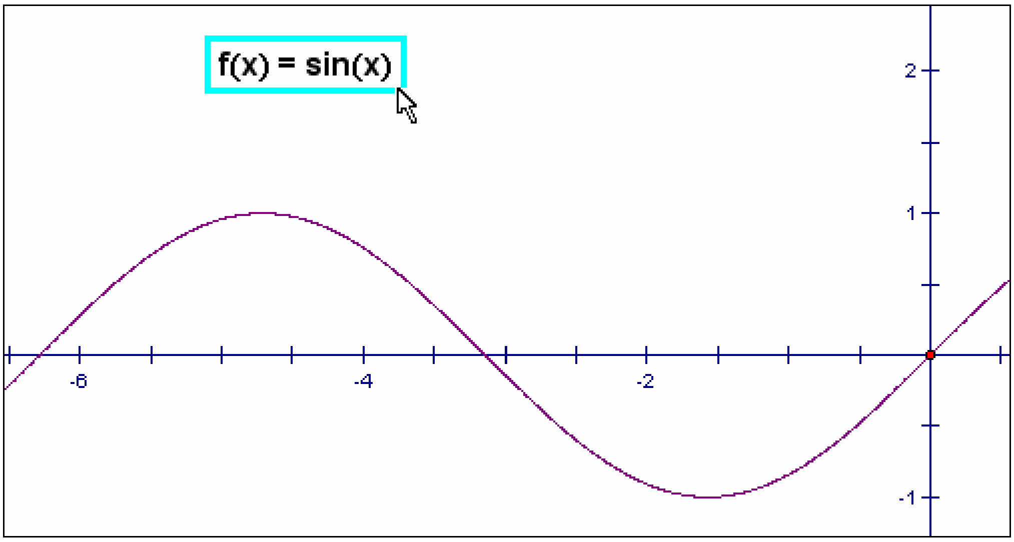
6. Normal: Normal: Select f(x) - Pt

To determine a normal we use its slope $-1/f'(x_0)$ and its foot $(x_0, f(x_0))$ as well as the displaced point:

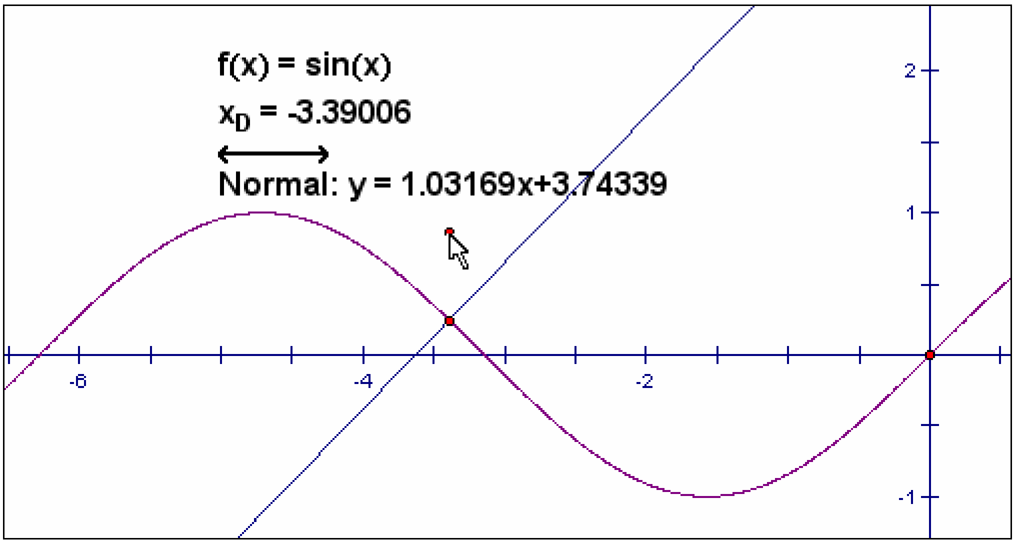
$$\left(x_0+1, f(x_0)-\frac{1}{f'(x_0)}\right).$$

When selecting the **normal tool** you must therefore select the function $f(x)$ and then point out the foot of the normal. Once you have selected the function the normal with its foot pops up on the screen at the graph point with the same x-coordinate as the mouse pointer together with the equation of the normal as well as the x-coordinate of the foot.

1 step: Point out the function:



2 step: Point out the foot of the normal:



Remark: The above normal has been constructed as a geometric line. If you need the normal as a graph of a first order polynomial (e.g. to be able to compute intersection points with the normal) you should plot a new function entering the symbolic equation for the normal instead: $f(x_0)-\frac{1}{f'(x_0)}\cdot(x-x_0)$.

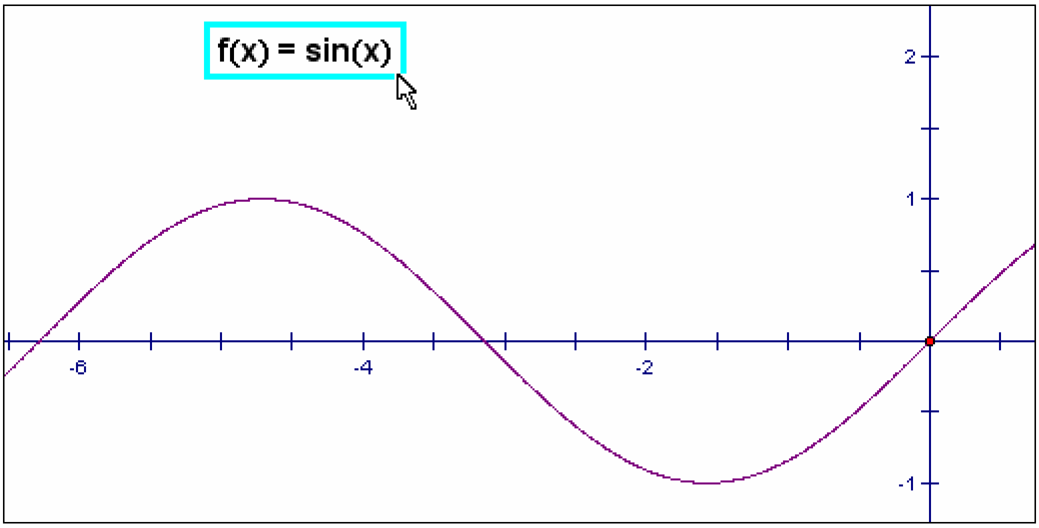
7. Curvature: Curvature: Select f(x) - Pt

To determine the center of curvature we use its coordinates:

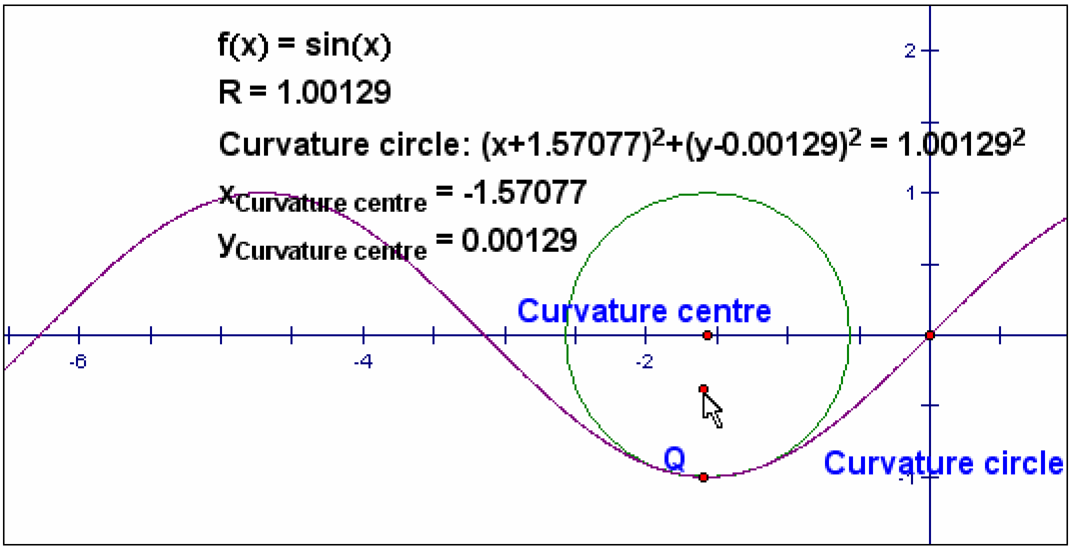
$$\left(x_0 - \frac{f'(x_0)}{f''(x_0)}\cdot\left(1+f'(x_0)^2\right),\, f(x_0)+\frac{1}{f''(x_0)}\cdot\left(1+f'(x_0)^2\right)\right)$$

When selecting the **curvature tool** you must therefore select the function $f(x)$ and then point out the point of contact for the curvature circle. Once you have selected the function the curvature circle with its centre pops up on the screen at the graph point with the same x-coordinate as the mouse pointer together with the equation of the circle as well as the coordinates of the centre and the radius of the circle.

1 step: Select the function:



2 step: Point out the point of contact:



Remark: The curvature centre is a point on the normal. In fact it is the limiting point of the intersection between neighboring normals.

Methods of integration: Some general remarks

To calculate numerically a definite integral

$$\int_a^b f(x)dx$$

we use Gauss' 3-point method which is as simple to implement as Simpson's 3-point method, but which is far more accurate. It is based upon the approximate summation:

$$\frac{h}{18}\cdot\left(5\cdot f\left(a+\frac{h}{2}\cdot\left(1-\sqrt{\frac{3}{5}}\right)\right)+8\cdot f\left(a+\frac{h}{2}\right)+5\cdot f\left(a+\frac{h}{2}\cdot\left(1+\sqrt{\frac{3}{5}}\right)\right)\right)$$

where h denotes the length of the interval, i.e. $h = b - a$.

According to how well behaved the function is you can obtain a reasonably value using only a limited number of partitions. If the function is smooth and slowly varying across the interval the value of the Gauss sum will typically stabilize after only 3-4 partitions. So for smooth functions 10 partitions will typically suffice. But if the function is rapidly varying or perhaps only piecewise smooth or have a vertical tangent you may need more partitions e.g. 100 partitions to stabilize the Gauss sum at its limit.

For these as well as other technical reasons the number of partitions is part of the tool! Before you can evaluate a definite integral numerically you must therefore first introduce a parameter *N* (or use a calculation or a measurement to compute the value of the parameter), which denotes the number of partitions of the domain of the integral.

This has both benefits and draw backs. Among the benefits are the following:

1. The user is reminded that the calculation is based upon a numerical approximation and not some kind of symbolic evaluation.
2. The user can check the accuracy of the calculation by doubling up the number of partitions. If the value of the integral approximation changes you must continue doubling up the parameter until the value is stable. In this way the experienced user can guarantee the accuracy of the result, while the inexperienced user can use a fairly high number, e.g. 100, and hope everything will still work out okay.
3. If the integral is used in further constructions e.g. a locus, it is important to keep down the number of partitions in order to keep the computation time low as well (the same holds for the number of points on the locus). But then it is clearly advantageous that you can select the number of partitions yourself, so that the computations is executed fast, while making it possible to check visually that the locus has stabilized on the screen.

Conclusion:

When you want to integrate you must therefore always remember to first introduce a parameter N denoting the number of partitions!

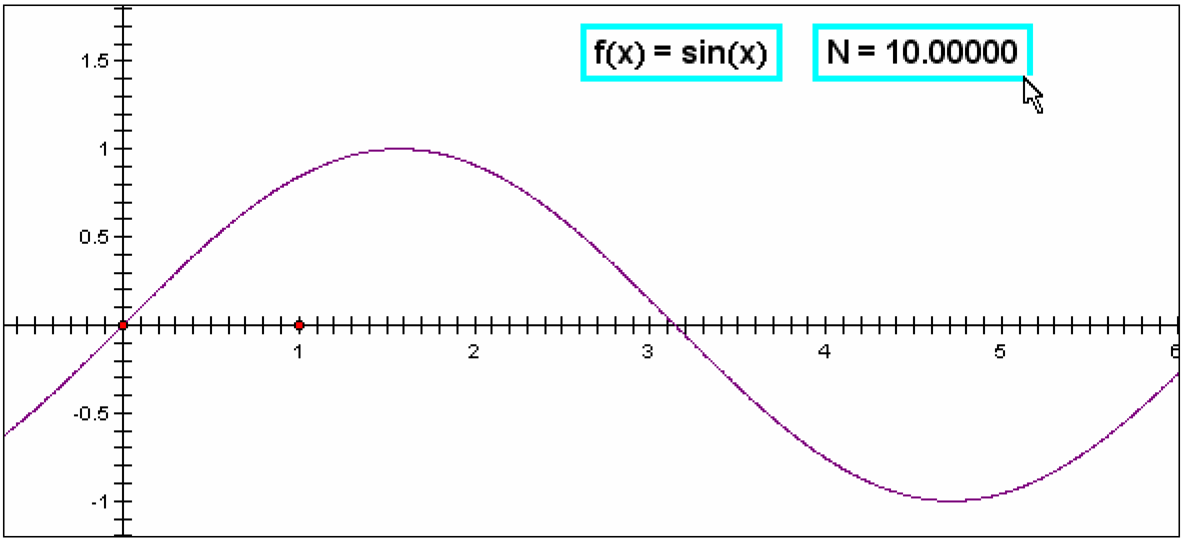
8. Integral with one function: Integral with one fct: Select f(x) - N - ptA - ptB

If e.g. you want to find the area of a standard region enclosed by a graph of a positive function *f*, the *x*-axis and two vertical lines $x = a$ and $x = b$, you must compute the definite integral:

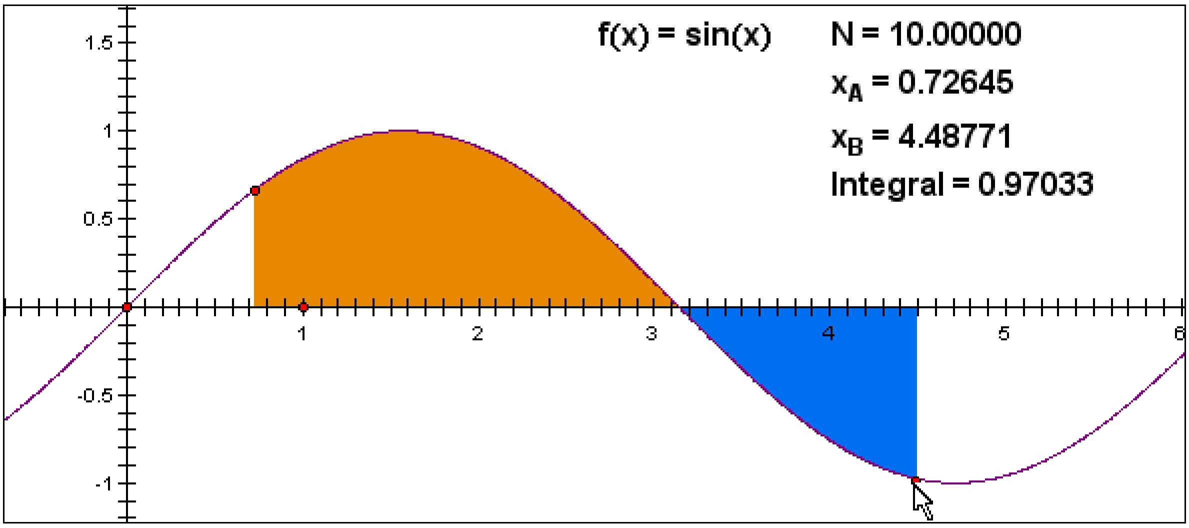
$$\int_a^b f(x)dx \ .$$

When selecting the **integration tool** you must therefore select a function as well as the number of partitions. Then you must point out the initial point *A* and the final point *B*. The region corresponding to the integration is hatched in a brown color for the part of the region above the *x*-axis and a blue color for the part below the *x*-axis. The area of the brown region contributes positively to the integral, while the blue region contributes negatively to the integral. The boundary values of the integration domain as well as the (approximate) integral are displayed. Remember that the initial point must lie to the left of the final point – unless you want to integrate backwards resulting in the opposite sign.

1 step: Select the function $f(x)$ as well as the number of partitions *N*:



2 step: Point out the initial and final boundary points :



Remark: The difference between the area of the brown region and the area of the blue region is thus 0.97033.

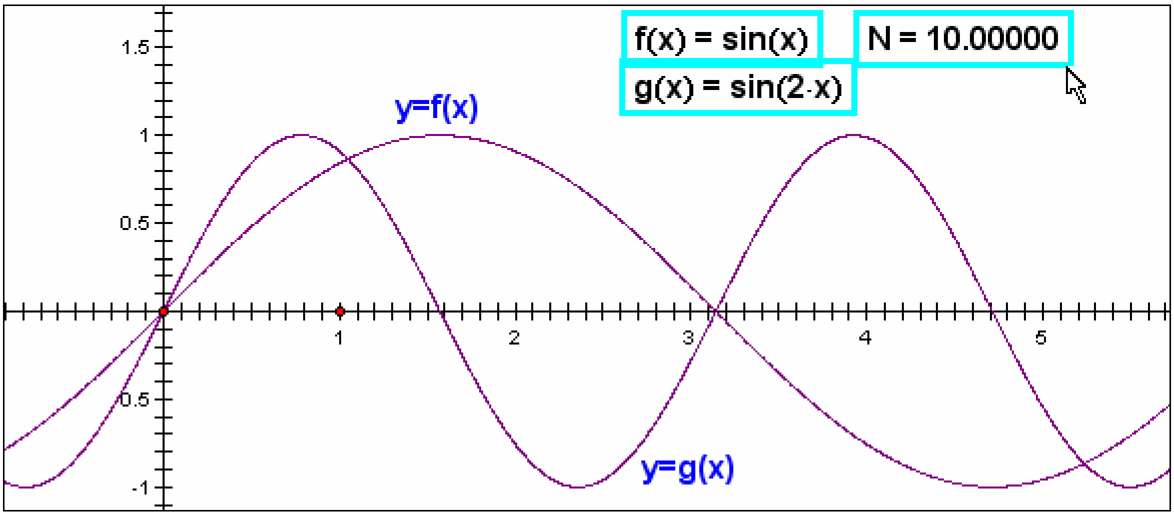
9. Integral between two fcts: Integral between two fcts: Select f(x) - g(x) - N - ptA - pt B

To find the area enclosed between the graphs of two functions f and g requires the computation of an integral of the form:

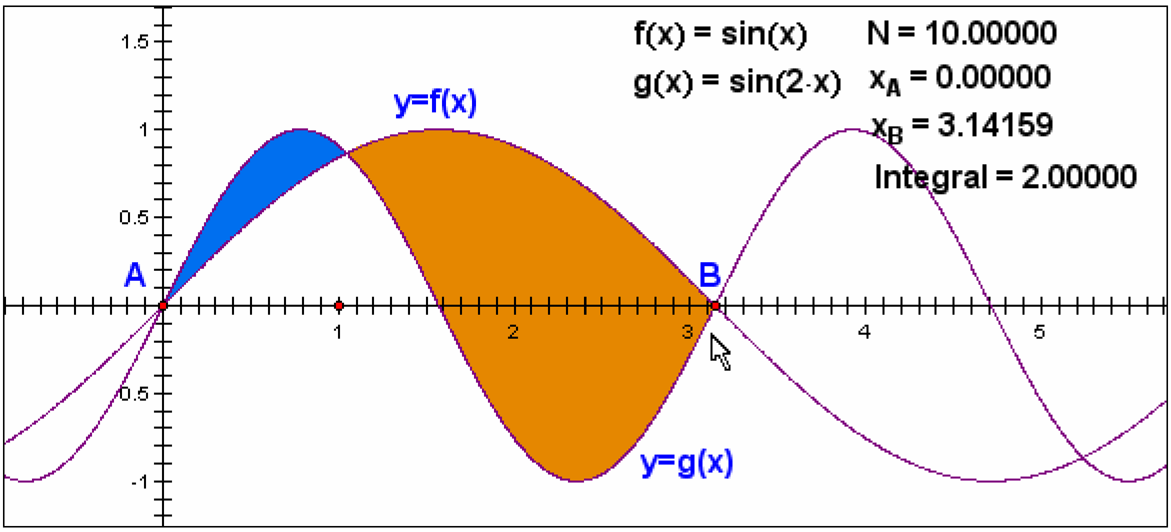
$$\int_a^b (f(x) - g(x))\, dx$$

When selecting the **Integration between two functions tool** you must therefore first select the two functions $f(x)$ and $g(x)$ as well as the number of partitions N. Then you must point out the initial point A and the final point B. The region corresponding to the integration is hatched in a brown color for the part of the region where the graph of f is the upper graph and a blue color for the part where the graph of g is the upper graph. The area of the brown region contributes positively to the integral, while the blue region contributes negatively to the integral. The boundary values of the integration domain as well as the (approximate) integral are displayed. Remember that the initial point must lie to the left of the final point – unless you want to integrate backwards resulting in the opposite sign.

1 step: Select $f(x)$ and $g(x)$as well as the number of partitions N:



2 step: Point out the initial and final boundary points:



Remark: The difference between the area of the large brown region and the small blue region is thus 2.

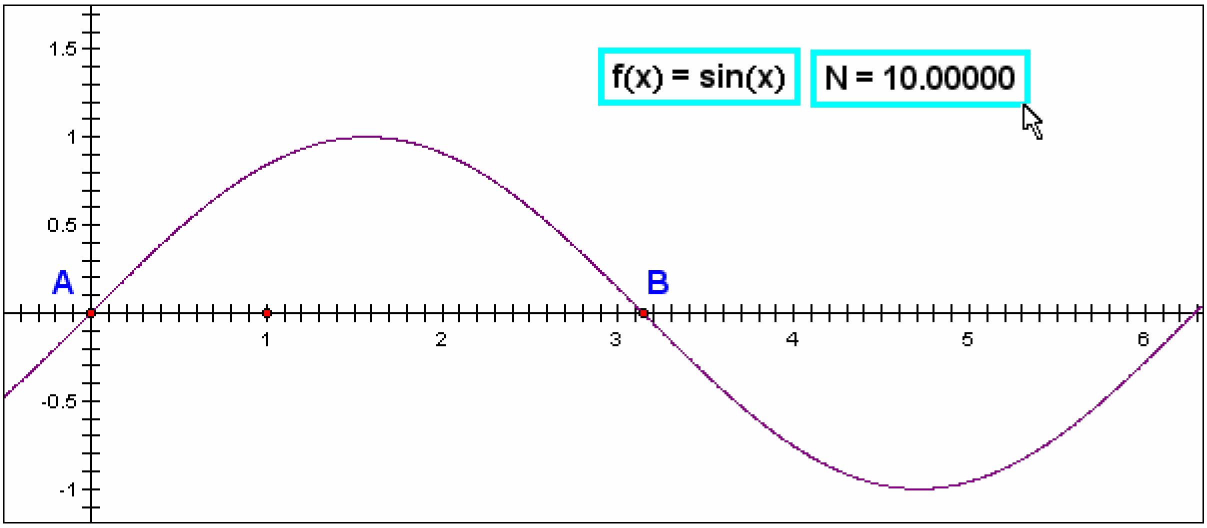
10. Arc length: Arc length: Select f(x) - N - ptA - ptB

To determine the arc length of an arc corresponding to a section of a graph of a differentiable function f, with the arc starting at $x = a$ and ending at $x = b$, you must compute the integral:

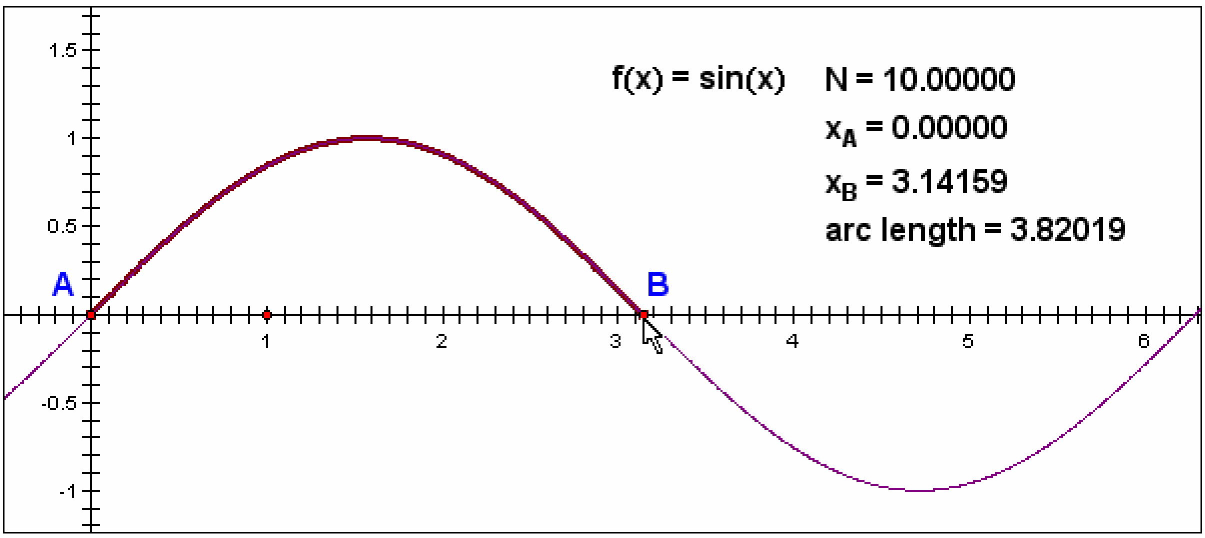
$$\int_a^b \sqrt{1+f'(x)^2}\; dx$$

When selecting the **arc length tool** you must therefore select the function $f(x)$ as well as the number of partitions N. Then you must point out the initial point A as well as the final point B for the arc. The section of the graph corresponding to the arc is colored brown. The boundary values of the integration domain as well as the (approximate) arc length are displayed. Remember that the initial point must lie to the left of the final point – unless you want to integrate backwards resulting in the opposite sign.

1 step: Select the function $f(x)$ as well as the number of partitions N:



2 step: Point out the initial and final boundary points:



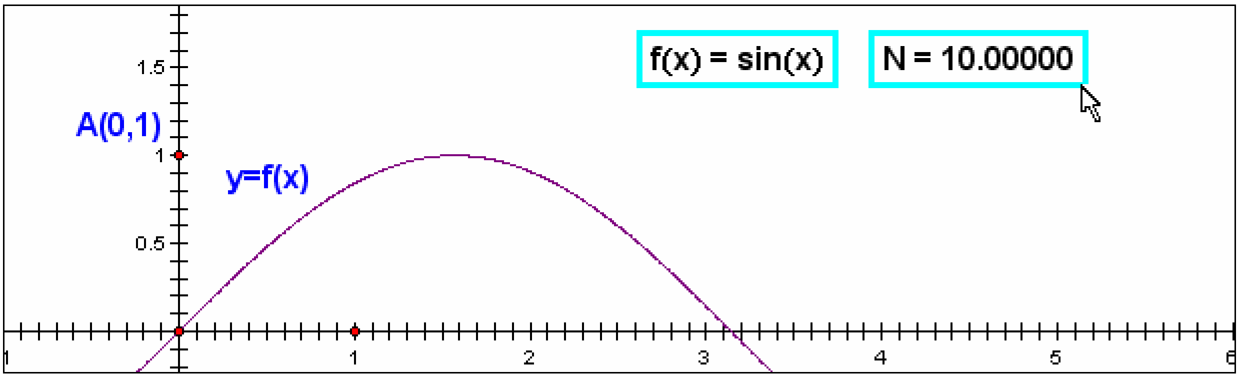
11. Antiderivative: Antiderivative: Select f(x) - N - x0 - endPt

To be able to draw the graph of the antiderivative (primitive) $F(x)$ associated with a given function $f(x)$ as a locus, you must first compute the integral:

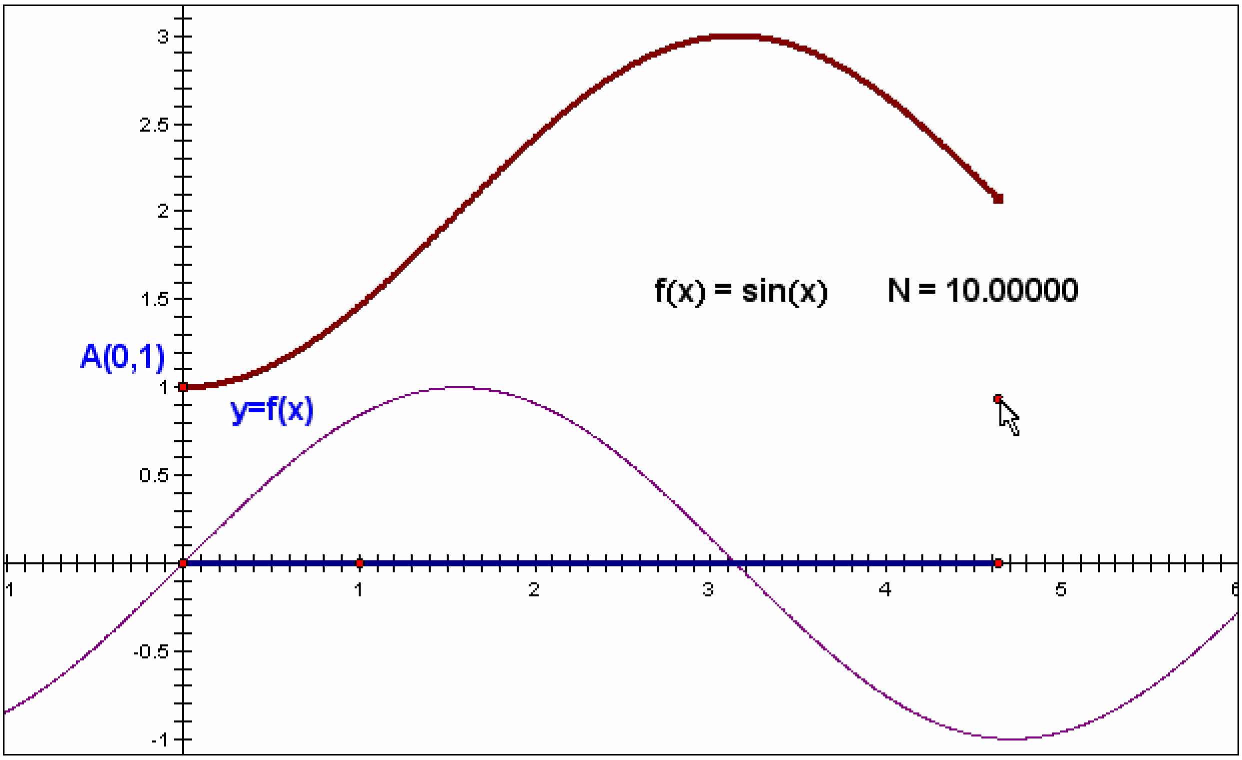
$$F(x) = y_0 + \int_{x_0}^{x} f(t)\,dt \ .$$

When selecting the **antiderivative tool** you must therefore first select the function $f(x)$ as well as the number of partitions N. Then you must select an initial point $(x_0,\ y_0)$ for the graph of the antiderivative as well as an end point for the domain of the antiderivative. The graph of the antiderivative is drawn as a locus together with the domain on the x-axis. If you want to draw the part of the graph prior to the initial point you simply choose the end point to the left of the initial point.

1 step: Select the function $f(x)$ as well as the number of partitions N:



2 step: Point out the initial and final boundary points:



Appendix:
The Gauss' 3-point method for numerical integration

Numerical integration is based upon an approximation of the integral with a weighted average of the form:

$$\int_a^b f(x)dx = \left(p_1 \cdot f(x_1) + p_2 \cdot f(x_2) + \ldots + p_n \cdot f(x_n)\right) \cdot (b-a)$$

There exist a variety of strategies for how to select the partition points $x_1, \ldots, x_n$ as well as the weight factors $p_1, \ldots, p_n$. To make a judgment of how accurate the method is we use the following family of test integrals

$$\int_{-1}^1 1 \cdot dx, \quad \int_{-1}^1 x \cdot dx, \quad \int_{-1}^1 x^2 \cdot dx, \quad \int_{-1}^1 x^3 \cdot dx, \quad \int_{-1}^1 x^4 \cdot dx, \quad \ldots$$

If the numerical integration technique works out the actual values of the first $n+1$ test integrals, i.e. up to the integral

$$\int_{-1}^1 x^n \cdot dx \ ,$$

then it turns out it will actually integrate any polynomial of degree n accurately over any interval $[a;b]$. The particular technique for numerical integration is then said to be of order $n+1$.

Simpson's method

Simpson's method belongs to a family of methods using an equidistant partition of the integration domain. Simpson's method is a 3-point method, i.e. it uses the partition points -1, 0 and 1. For reasons of symmetry it uses symmetrical weights, i.e. the Simpson method is based upon a weighted average of the form

$$(p \cdot f(-1) + q \cdot f(0) + p \cdot f(1)) \cdot 2 \ .$$

The symmetry takes care of all the test integrals of odd order. As regards the test integrals of even order we obtain:

$$2 = \int_{-1}^1 1 \cdot dx \approx \left(p + q + p\right) \cdot 2$$

$$\tfrac{2}{3} = \int_{-1}^1 x^2 \cdot dx \approx \left(p \cdot (-1)^2 + q \cdot 0^2 + p \cdot 1^2\right) \cdot 2 = 2p \cdot 2$$

It follows that we must select the following values of the weight factors p and q:

$$2 \cdot p + q = 1 \text{ and } 2 \cdot p = \tfrac{1}{3} \ .$$

The first condition simply states that the sum of the weight factors as expected must be 1. The other condition tells us that $p = 1/6$ and consequently $q = 2/3$. Simpson's method is thus based upon the following approximation:

$$\int_a^b f(x) \cdot dx \approx \left(\frac{1}{6} \cdot f(a) + \frac{2}{3} \cdot f(\frac{a+b}{2}) + \frac{1}{6} \cdot f(b)\right) \cdot (b-a) = \frac{f(a) + 4 \cdot f(\frac{a+b}{2}) + f(b)}{6} \cdot (b-a)$$

As we see Simpson's method is a **fourth order method** i.e. it calculates integrals of cubic polynomials accurately.

As an example of Simpson's method for numerical integration we look at the following simple definite integral:

$$2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x)\cdot dx \approx \frac{\cos(-\frac{\pi}{2}) + 4\cdot\cos(0) + \cos(\frac{\pi}{2})}{6}\cdot\pi = \frac{0+4\cdot 1+0}{6}\cdot\pi = \frac{2\pi}{3} = 2.094...$$

The relative error is thus approximately 5%. Notice by the way that Simpson's method corresponds to replacing π with 3!

Gauss' method

Gauss improved Simpson's method by using his freedom to choose a more appropriate set of partition points. Like Simpson's method Gauss' method is a 3-point method with a symmetrical choice of partition points, i.e. Gauss' method uses the partition points $-x$, 0 and x. For reasons of symmetry the weight factors are also symmetrical, i.e. Gauss' method is based upon a weighted average of the following form:

$$(p\cdot f(-x) + q\cdot f(0) + p\cdot f(x))\cdot 2$$

As before the symmetry takes care of all tests integrals of odd order. As regards the test integrals of even order we obtain:

$$2 = \int_{-1}^{1} 1\cdot dx \approx \left(p+q+p\right)\cdot 2 = (2\cdot p+q)\cdot 2$$

$$\tfrac{2}{3} = \int_{-1}^{1} x^2\cdot dx \approx \left(p\cdot(-x)^2 + q\cdot 0^2 + p\cdot x^2\right)\cdot 2 = 2p\cdot x^2\cdot 2$$

$$\tfrac{2}{5} = \int_{-1}^{1} x^4\cdot dx \approx \left(p\cdot(-x)^4 + q\cdot 0^4 + p\cdot x^4\right)\cdot 2 = 2p\cdot x^4\cdot 2$$

It follows that the weight factors p and q must satisfy the conditions:

$$2\cdot p + q = 1,\quad 2\cdot p\cdot x^2 = \tfrac{1}{3} \quad\text{and}\quad 2\cdot p\cdot x^4 = \tfrac{1}{5}\ .$$

The first condition simply states that the sum of the weight factors as expected must be 1. The other conditions simplify to

$$p\cdot x^2 = \tfrac{1}{6} \quad\text{and}\quad p\cdot x^4 = \tfrac{1}{10}$$

By division we obtain the equation $x^2 = \frac{\frac{1}{10}}{\frac{1}{6}} = \frac{3}{5} \Rightarrow x = \sqrt{\frac{3}{5}}$. Substituting this into the first condition we similarly get $p\cdot\frac{3}{5} = \frac{1}{6} \Rightarrow p = \frac{5}{18}$. This leads to $q = \frac{8}{18} = \frac{4}{9}$. Gauss' method is thus based upon the following approximation:

$$\int_{-1}^{1} f(x)\cdot dx \approx \frac{5\cdot f(-\sqrt{\frac{3}{5}}) + 8\cdot f(0) + 5\cdot f(\sqrt{\frac{3}{5}})}{18}\cdot 2$$

When applied to an arbitrary domain of integration $[a;b]$ we use that 0 corresponds to the centre of the domain, i.e.

$$\frac{a+b}{2},$$

whereas 1 corresponds to the radius of the domain, i.e.

$$\frac{b-a}{2}.$$

Thus the above formula is extended in the following way:

$$\int_a^b f(x)\cdot dx = \frac{5\cdot f(\frac{a+b}{2}-\frac{b-a}{2}\cdot\sqrt{\frac{3}{5}})+8\cdot f(\frac{a+b}{2})+5\cdot f(\frac{a+b}{2}+\frac{b-a}{2}\cdot\sqrt{\frac{3}{5}})}{18}\cdot(b-a)$$

It follows that Gauss' 3-point method is a **sixth order method**, i.e. it calculates integrals of quintic polynomials accurately.

As an example of Gauss' method for numerical integration we look at the same simple definite integral as before:

$$2=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos(x)\cdot dx \approx \frac{5\cdot\cos(-\frac{\pi}{2}\cdot\sqrt{\frac{3}{5}})+8\cdot\cos(0)+5\cdot\cos(\frac{\pi}{2}\cdot\sqrt{\frac{3}{5}})}{18}\cdot\pi$$
$$=\frac{5\cdot\cos(\frac{\pi}{2}\cdot\sqrt{\frac{3}{5}})+4}{9}\cdot\pi=2.0013889...$$

The relative error is thus below 1‰.

As claimed Gauss' method is thus much more accurate than Simpson's method. Despite the somewhat clumsy choice of partition points it is therefore clearly preferable to use Gauss' method for numerical integration. There are even theoretical advantages: Since Gauss' method avoids using the boundary points for the estimation it can also handle integrands with vertical asymptotes at the boundary.