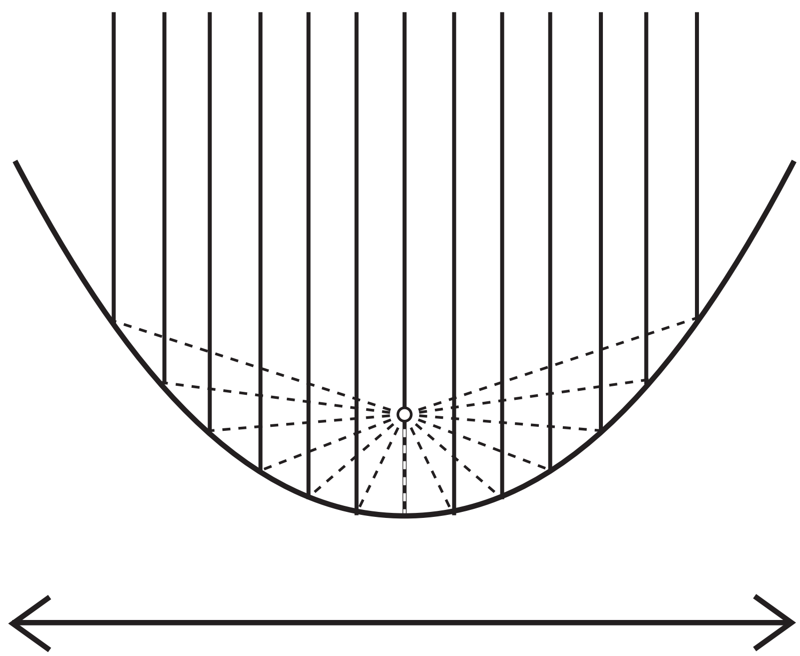
4

Quadratic Functions



Parabolas in Vertex Form

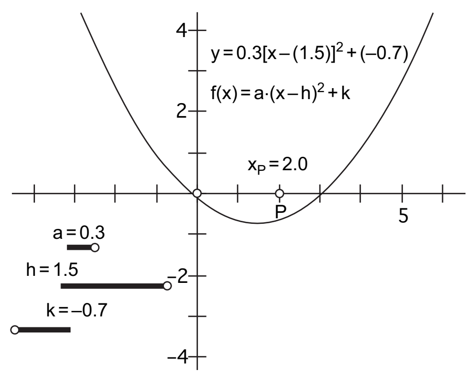
Things with *bilateral symmetry,* such as the human body, have parts on the sides that come in pairs (such as ears and feet) and parts down the middle that there's just one of (such as the nose and belly button). Parabolas are the same way. Points on one side have corresponding points on the other. But one point is unique: the vertex. It's right in the middle, and, like your nose, there's just one of it. Not surprisingly, there's a common equation form for parabolas that relates to this unique point.

SKETCH AND INVESTIGATE

1. Open **Vertex Form.gsp.** You'll see an equation in the form $y = a(x - h)^2 + k$, with a, h, and k filled in, and sliders for a, h, and k. Adjust the sliders (by dragging the points at their tips) and watch the equation change accordingly. There's no graph yet; you'll create that in the next step.

To enter *a, h,* and *k,* click their measurements in the sketch. To enter *x,* click on the *x* in the Calculator keypad.

2. Choose **Graph | Plot New Function.** The New Function dialog box appears. If necessary, move it so that you can see the measurements of a, h, and k. Enter $a*(x-h)\wedge 2 + k$ and click OK. Sketchpad plots the function for the current values of a, h, and k.



Notice point P on the x-axis. The measurement x_p is the x-coordinate of this point. You'll now plot on the parabola a point that has the same x-coordinate.

Choose **Measure | Calculate.** Click on the function equation from step 2. Then click on x_p to enter it, and click OK.

3. Calculate $f(x_p)$, the value of the function f evaluated at x_p. You'll see an equation for $f(x_p)$, the value of the function f evaluated at x_p.

4. Select, in order, x_p and $f(x_p)$. Choose **Graph | Plot As (x, y).** This command plots a point on the parabola. Label the new point Q. Drag P if you can't see Q.

Q1 Using paper and pencil or Sketchpad's Calculator, show that the coordinates of point Q satisfy the parabola's equation. If the numbers are a little off, explain why this might be.

EXPLORE FAMILIES OF PARABOLAS

By dragging point P, you're exploring how the variables x and y vary along one particular parabola with particular values for a, h, and k. For the rest of this activity, you'll change the values of a, h, and k, which will change the parabola itself, allowing you to explore whole families of parabolas.

Q2 Adjust slider a and observe the effect on the parabola.

Summarize the role of a in the equation $y = a(x - h)^2 + k$. Be sure to discuss its sign, its magnitude, and anything else that seems important.

Q3 Changing a appears to change all the points on the parabola but one, the vertex. Change the values of h and k; then adjust a again, focusing on where the vertex appears to be. How does the location of the vertex relate to the parameters h and k?

Q4 Adjust the sliders for h and k. Describe how changing h transforms the parabola. How does that compare to the way that changing k transforms it?

To use **Graph | Plot As (x, y),** first select exactly two measurements to be used as the values for *x* and *y*.

5. Plot the vertex of your parabola using the **Graph | Plot As (x, y)** command.

Q5 Write the equation in vertex form $y = a(x - h)^2 + k$ for each parabola described below. As a check, adjust the sliders so that the parabola is drawn on the screen.

Note: In this activity, the precision of measurements has been set to one decimal place. It's important to be aware of this and to check your answers by hand, in addition to adjusting the sliders in the sketch.

a. vertex at $(1, -1)$; y-intercept at $(0, 4)$

b. vertex at $(-4, -3)$; contains the point $(-2, -1)$

c. vertex at $(5, 2)$; contains the point $(1, -6)$

d. same vertex as the parabola $-3(x - 2)^2 - 2$; contains the point $(0, 6)$

e. congruent to the parabola $4(x + 3)^2 - 1$; vertex at $(-1, 3)$

Q6 The axis of symmetry is the line over which a parabola can be flipped and still look the same. What is the equation of the axis of symmetry for the parabola $y = 2(x - 3)^2 + 1$? What is the general equation for the line of symmetry corresponding to the parabola $y = a(x - h)^2 + k$?

Q7 Just as your right ear has a corresponding ear across your body's axis of symmetry, all points on a parabola (except the vertex) have corresponding points across its axis of symmetry.

The point $(5, 9)$ is on the parabola $y = 2(x - 3)^2 + 1$. What is the corresponding point across the axis of symmetry?

EXPLORE MORE

Q8 Move point Q (s, t) to the right half of the parabola $y = a(x - h)^2 + k$. What is the corresponding point Q' across the axis of symmetry, in terms of s and t? If (s, t) were on the left half of the parabola, what would the answer be?

Q9 Use **Construct | Perpendicular Line** to construct the axis of symmetry of your parabola. Then use **Transform | Reflect** to reflect point Q across the axis of symmetry. Measure the coordinates of the new point, Q'. Are they what you expected?

Parabolas in Vertex Form

Objective: Students vary parameters on a quadratic function in vertex form and create a graph satisfying specific conditions.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students need to understand the basic idea of a function and the role that the variables x and y play in the equation and graph of a function. Solving simple linear equations for one unknown after substituting given values for other unknowns is also part of this activity.

Sketchpad Level: Easy

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Vertex Form.gsp**) or Whole-Class Presentation (use **Vertex Form Present.gsp**)

Related Activities: The activity Exploring Parabolas in Vertex Form covers much of the same material, but in a more open-ended way.

You may wish to draw attention to the similarities between this equation form for the parabola and the point-slope form of a line (as well as the formula used in the Absolute Value Functions activity, if students have done that). A good topic of discussion is whether there's a notion of slope with parabolas. If so, how is it different from the slope of a line? If not, what does a control?

SKETCH AND INVESTIGATE

Q1 There are many possible answers.

This is a good calculator activity for substituting an x-value into the right side of a function to determine a y-value. Any small discrepancy is due to Sketchpad's rounding (depending on the Precision settings in Preferences).

EXPLORE FAMILIES OF PARABOLAS

Q2 If a is positive, the parabola opens upward; if a is negative, the parabola opens downward. The larger the absolute value of a, the narrower the parabola. The closer a is to zero, the wider the parabola.

Q3 The coordinates of the vertex are (h, k).

Q4 The parabola moves right and left as h changes (right as h gets bigger, left as it gets smaller). The parabola moves up and down as k changes (up as k gets bigger, down as it gets smaller).

Q5
a. $y = 5(x - 1)^2 - 1$
b. $y = 0.5(x + 4)^2 - 3$
c. $y = -0.5(x - 5)^2 + 2$
d. $y = 2(x - 2)^2 - 2$
e. $y = 4(x + 1)^2 + 3$

It's very important that students find the equations of these parabolas using paper-and-pencil calculations and use Sketchpad to check their answers. Slider accuracy may account for small differences.

Q6 $x = 3$; $x = h$

Q7 $(1, 9)$

EXPLORE MORE

Q8 Since $x = h$ is the axis of symmetry, $Q'(2h - s, t)$ is the reflected image of $Q(s, t)$. This relationship will hold no matter which side of the parabola point Q is on.

Q9 The coordinates of points Q and Q' should be consistent with the relationship described above.

Before beginning this presentation, it would be a good idea to review the fact that graphs of quadratic functions are parabolas. Also, you can expand some vertex form quadratic functions to show that they are in fact quadratic.

1. Open **Vertex Form Present.gsp.** Drag the sliders one at a time so that students can see how they connect with the function at the top of the screen.
2. Press the *Show Graph* button.

Q1 Ask the class what will happen when you drag the slider for parameter a. After giving them sufficient time for discussion, drag the slider. Be sure to show positive and negative values, and give some attention to the special case of $a = 0$.

Q2 Move on to the sliders for parameters h and k. Again, ask students to make predictions before you actually change any values.

Q3 What are the coordinates of the vertex of this parabola? (The coordinates are (h, k).)

3. Select in order parameters h and k. Choose **Graph | Plot As (x, y).** Drag the sliders one at a time to show that the new plotted point stays at the vertex.

Q4 Ask students how we can change this parabola so that the vertex is at $(0, 4)$ and the graph contains the point $(1, -1)$. The most obvious way is to alternate dragging each of the three sliders until the desired result appears. Use whatever method the class suggests.

Q5 Hide the graph. Ask students how to put the vertex at $(-4, -3)$ and make the graph go through point $(-2, -1)$. Knowing the vertex should tell them that $h = -4$ and $k = -3$. Show them how to substitute these values into the equation along with $x = -2$ and $y = -1$. Then solve for a ($a = 0.5$). Press *Show Graph* to test your solution. Try a few other examples.

Q6 All parabolas have an axis of symmetry. Where is the axis of this parabola? What is the equation of the axis? (The axis is a vertical line through the vertex, and its equation is $x = h$.)

4. Select the vertex and the x-axis. Choose **Construct | Perpendicular Line.**
5. Construct a point on the parabola.

Q7 Ask, "If we reflect this point across the axis of symmetry, where will it fall?" (The reflected image should also be on the parabola.)

6. Double-click the axis of symmetry. Select the point on the parabola. Choose **Transform | Reflect.** Drag both the point and the sliders to show that this works in all cases.

Exploring Parabolas in Vertex Form

This activity is an open-ended exploration of quadratic functions expressed in vertex form. Play with the various pages in **Explore Vertex Form.gsp** and let your curiosity be your guide. The questions below are meant to point you toward interesting areas of inquiry, but you or your teacher may decide that other questions are more interesting or relevant. Happy exploring!

EXPLORE

Open **Explore Vertex Form.gsp.** Use the link buttons to navigate from page to page. The various pages are described below.

$y = x^2$ (**basic parabola**)	The vanilla parabola, the base model: no frills, no coefficients, nothing added. Understand this parabola and you're on your way to understanding them all.
$y = ax^2$ (**slider**) $y = ax^2$ (**direct**)	What happens when you multiply x^2 by a constant? Find out on these two pages. On the first, the parameter a is controlled by a slider. On the second, it's controlled directly by a point on the parabola. The underlying math is the same for both.
$y = ax^2 + k$ (**slider**) $y = ax^2 + k$ (**direct**)	What happens when you add a constant k to ax^2? Again, a and k are controlled with sliders on one page and directly with points on the parabola on the other, but the math is the same.
$y = a(x - h)^2 + k$ (**slider**) $y = a(x - h)^2 + k$ (**direct**)	What happens when you now subtract a constant h from the x in $y = ax^2 + k$? Find out here.

QUESTIONS TO PONDER, DISCUSS, OR WRITE ABOUT

Q1 Name all of the points on $y = x^2$ that are visible in the current window and that have integers for both x- and y-coordinates. Without scrolling, name four other points on $y = x^2$ with integer coordinates.

Q2 What can you say about the equation of a parabola if its vertex is at the origin?

Q3 What kind of symmetry do all parabolas in the family $y = ax^2$ exhibit? Why do they have this symmetry?

Q4 If the point (c, d) is on the parabola $y = x^2$, name one other point that must be on the parabola.

Q5 How does the sign of a affect the parabolas on these pages?

Q6 What do the various graphs look like if $a = 0$?

Q7 How is the coefficient a in these equations similar to the coefficient m in $y = mx$, $y = mx + b$, and $y = m(x - h) + k$? How is it different?

Q8 The third page shows $y = ax^2$ and lets you can control a with a drag control located one unit to the right of the axis of symmetry (the y-axis). Why does it make sense for the drag control to be there?

Q9 The axis of symmetry is the line a parabola can be flipped over and still look the same. The axis of symmetry for $y = x^2$ is $x = 0$ (the y-axis). What is the equation of the axis of symmetry for parabolas in the family $y = ax^2$? $y = ax^2 + k$? $y = a(x - h)^2 + k$?

Q10 Transformations used in geometry include translations, rotations, dilations, reflections, stretchs, shrinks, and shears. Look up these terms, then describe which apply to the transformations to parabolas you observe when dragging the sliders for a, h, and k.

EXPLORE MORE

Use commands from the Graph and/or Construct menus to construct the axis of symmetry of the parabola on one of the last two pages. Then use commands from the Transform menu to reflect point P across this axis.

Objective: Students use a prepared sketch to explore the connections between parameters of quadratic functions and their graphs.

Student Audience: Algebra 1/Algebra 2/Geometry

Prerequisites: None in particular. Some familiarity with parabolas and with graphical transformations would help.

Sketchpad Level: Easy. Students manipulate sliders in a pre-made sketch.

Activity Time: 20–40 minutes. This depends on how many questions are explored, whether answers are written or discussed, how detailed the answers are, and so on.

Setting: Paired/Individual Activity (use **Explore Vertex Form.gsp**) or Whole-Class Presentation (use **Explore Vertex Form Present.gsp**)

Related Activities: The activity Parabolas in Vertex Form is concerned with the same subject matter as this activity, but approaches it in a much different way. That activity uses a more traditional, step-by-step approach, and it focuses exclusively on the vertex form. This activity uses a more open-ended approach and builds toward the vertex form from much more basic parabola forms. Which approach is more appropriate depends on the particular students and the particular class involved.

QUESTIONS TO PONDER, DISCUSS, OR WRITE ABOUT

Q1 The integer points visible in the sketch window are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(2, 4)$, $(-2, 4)$, $(3, 9)$, and $(-3, 9)$. Four other integer points not visible but on the parabola are $(4, 16)$, $(-4, 16)$, $(5, 25)$, and $(-5, 25)$. (Many other answers are possible for the second part of this question.)

Q2 It must be of the form $y = ax^2$. In vertex form, a parabola with vertex $(0, 0)$ is written as $y = a(x - 0)^2 + 0$, which simplifies to $y = ax^2$.

Q3 They are all symmetrical across the y-axis. This is because the square of a number is the same as the square of its opposite. For example, if $y = 2x^2$, both $(3, 18)$ and $(-3, 18)$ are on the graph because $3^2 = (-3)^2$. These two points are reflections of each other across the y-axis. Functions that have this property, $f(-x) = f(x)$, are called *even* functions.

Q4 Point $(-c, d)$ is also on the parabola.

Q5 If a is positive, the parabola opens upward. If a is negative, it opens downward. As the magnitude of a gets bigger, the parabola takes on narrower form.

Q6 They become horizontal lines.

Q7 Coefficient m controls the slopes of lines and a controls something for parabolas that's very much like slope: how narrow (steep) or wide (flat) a parabola is. The sign of m determines whether the graph rises or falls as you move to the right. The sign of a determines whether the graph rises or falls as you move horizontally away from the vertex.

Q8 It makes sense because the y-coordinate at $x = 1$ is a. To see this, just substitute 1 for x in $y = ax^2$. You get $y = a$.

Q9 The equation of the axis of symmetry for both $y = ax^2$ and $y = ax^2 + k$ is $x = 0$. The equation of the axis of symmetry for $y = a(x - h)^2 + k$ is $x = h$.

Q10 Dragging h and k results in horizontal and vertical translations, respectively. Dragging a stretches the graph vertically.

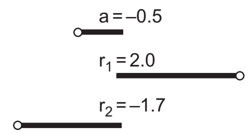
EXPLORE MORE

One way to do this is to construct a vertical line through the vertex. The vertex is already plotted on the direct-control page. To plot the vertex on the slider-control page, select h and k in that order and choose **Graph | Plot As (x, y).** Now select the vertex and the x-axis and choose **Construct | Perpendicular Line.**

Parabolas in Factored Form

If you use a parabola to model a thrown ball, you might want to know exactly where it hits the ground. Or if you model a profit function with a parabola, you might be interested in the *break-even point*—the cutoff between profitability and loss. In both cases you're interested in the *roots* of the equation—where the function's value is zero. In this activity you'll explore a quadratic form that's based on the roots.

SKETCH AND INVESTIGATE



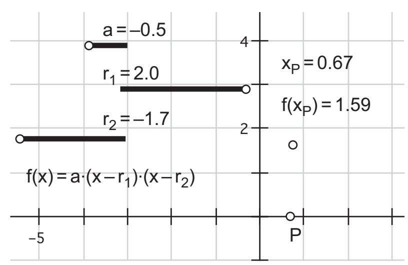
1. Open **Factored Form.gsp.** This sketch contains three sliders (a, r_1, and r_2). Adjust each slider by dragging the point at its tip. Observe how the values change as you drag.

First you'll create a function, evaluate it for an input variable, and plot the resulting point.

2. Use the slider values to define the function $f(x) = a(x - r_1)(x - r_2)$. To do so, choose **Graph | New Function.** The New Function dialog box appears. If necessary, move it so that you can see the measurements of a, r_1, and r_2.

To enter a, r_1, and r_2, click their measurements in the sketch. To enter x, click the x key in the dialog box.

3. Enter $a*(x-r_1)*(x-r_2)$ and click OK. Sketchpad creates the function.
4. Measure the x-coordinate of point P by choosing **Measure | Abscissa.**
5. Calculate $f(x_P)$, the value of function f evaluated at x_P. To do so, choose **Measure | Calculate.** Click on the function object $f(x)$ and then on measurement x_P. Finally click OK. The value of the function appears.
6. To plot the point, select x_P and $f(x_P)$ and choose **Graph | Plot As (x, y).**



To turn tracing on or off, select the plotted point and choose **Display | Trace Plotted Point.**

Q1 Drag P back and forth along the x-axis to change the input variable for the function. How does the plotted point behave? Turn on tracing for the plotted point to better observe its behavior. What shape does this function trace out?

Traces are temporary in Sketchpad. Instead of using traces, you'll now make the graph permanent so you can compare the graphs of different functions.

7. Turn off tracing for the plotted point. Then choose **Display | Erase Traces** and drag *P* to make sure the traces no longer appear.
8. Plot a permanent graph of the function by selecting the function and choosing **Graph | Plot Function.** Drag *P* back and forth to make sure that the graph really corresponds to the path of the plotted point.

EXPLORING FAMILIES OF PARABOLAS

By dragging point *P*, you explored how x and y vary for *one particular function* with specific values of a, r_1, and r_2. Now you'll change the values of a, r_1, and r_2, which *changes the function itself,* allowing you to explore whole families of parabolas.

Q2 Adjust slider a and describe its effect on the parabola. Discuss the effect of a's sign (whether it's positive or negative), its magnitude (how big or small it is), and anything else that seems important.

Q3 Dragging a appears to change all the points on the parabola but two: the x-intercepts of the parabola (the roots). Adjust all three sliders and observe the effect that each has on the x-intercepts. How are the locations of the x-intercepts related to the values of the sliders?

Q4 Adjust slider r_1. What happens to the parabola as r_1 changes? What happens as r_2 changes?

Q5 Adjust the sliders so that $r_1 = r_2$. Describe the resulting parabola.

Figure out the equations using pencil and paper only. Once you think you have the equation, you can use the sketch to check your result.

Q6 For each description below, write an equation for a parabola in factored form $f(x) = a(x - r_1)(x - r_2)$. Check your answers by adjusting the sliders.

a. x-intercepts at $(-4, 0)$ and $(6, 0)$; vertex at $(1, -1)$

b. x-intercepts at $(-5, 0)$ and $(1, 0)$; contains the point $(3, 32)$

c. x-intercepts at $(0, 0)$ and $(-3, 0)$; contains the point $(2, 3)$

d. same x-intercepts as $y = 2(x - 3)(x + 1)$; contains the point $(0, -3)$

e. same shape as $y = 2(x - 3)(x + 1)$; x-intercepts at $(-4, 0)$ and $(1, 0)$

Q7 You throw a baseball and it flies in a parabolic path across a field. If the ball reaches its apex (highest point) 60 feet away from you, and the apex is 40 feet above the ground, how far away from you will the ball land? What is the equation of the ball's flight in factored form? (Assume that the ball starts at the point $(0, 0)$.)

EXPLORE MORE

Q8 When you have a parabola in the form $f(x) = a(x - h)^2 + k$, it's easy to find its vertex, but harder to find its roots. The opposite is true with the form $f(x) = a(x - r_1)(x - r_2)$. Explain what you do know about the vertex of parabolas in this form. Can you write an expression for the x-coordinate of the vertex in terms of r_1 and r_2? The y-coordinate?

9. Use the expressions you just wrote for the coordinates of the vertex to plot the vertex. If you do this properly, the plotted point will remain at the vertex regardless of how you drag the sliders. You can use Sketchpad's Calculator to calculate the x- and y-coordinates of the vertex using the expressions you found in the previous question. You can then select the two calculations and choose **Graph | Plot As (x, y).**

Q9 If you successfully plotted the vertex, try this: Turn on tracing for the vertex by choosing **Display | Trace Plotted Point.** Then adjust slider r_1. What shape does the vertex trace? Can you write an equation for this curve in terms of a and r_2?

Parabolas in Factored Form

Objective: Students plot the graph of a quadratic function in factored form, investigate the relationship between this form and its graph, and use their observations to create functions from various descriptions of their graphs.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should have some experience with graphing quadratic equations. It's not necessary for them to be familiar already with the factored form.

Sketchpad Level: Intermediate. Students create and evaluate a function, plot and trace a point, and plot a function.

Activity Time: 40–50 minutes. The activity is shorter if you use the alternate sketch **Factored Form 2.gsp** and start on the second page of the activity.

Setting: Paired/Individual Activity (use **Factored Form.gsp**) or Whole-Class Presentation (use **Factored Form Present.gsp**)

Related Activities: Parabolas in Standard Form, Parabolas in Vertex Form, Changing Quadratic Function Forms

It's important that students understand that a parabola remains a parabola regardless of the form its equation is written in. Any parabola's equation can be written in many different forms. The forms we study are the ones that are especially simple or that convey special information.

The factored form of a parabola is different from the standard form and the vertex form, because parabolas that don't cross or touch the x-axis cannot be written in factored form. When you summarize this activity, you might discuss the existence of roots and the possibility of categorizing parabolas based on whether they have 0, 1, or 2 distinct real roots. If students have studied the discriminant, this would be a good connection to make.

SKETCH AND INVESTIGATE

Q1 The plotted point follows a curved, parabolic path. The exact shape (narrow or wide, opening up or down) depends on the values of the sliders.

EXPLORING FAMILIES OF PARABOLAS

Q2 If a is positive, the parabola opens upward; if a is negative, the parabola opens downward. The larger the absolute value of a, the narrower the parabola. The closer a is to zero, the wider the parabola.

Q3 When you drag a, the two x-intercepts remain fixed. One x-intercept exactly matches the value of slider r_1, while the other exactly matches the value of slider r_2.

Q4 Dragging either r slider changes the x-intercept corresponding to that slider.

Q5 A parabola with two equal roots has its vertex on the x-axis. Such roots are called *double roots.*

Q6
a. $y = 0.04(x + 4)(x - 6)$

b. $y = 2(x + 5)(x - 1)$

c. $y = 0.3(x - 0)(x + 3)$ or $y = 0.3x(x + 3)$

d. $y = 1(x - 3)(x + 1)$ or $y = (x - 3)(x + 1)$

e. $y = 2(x + 4)(x - 1)$

It is very important that students find the equations of these parabolas using paper-and-pencil calculations and use Sketchpad to check their answers.

Q7 The ball will land 120 feet away. The equation is $y = \left(\frac{-1}{90}\right)(x - 0)(x - 120)$ or $y = \left(\frac{x}{90}\right)(x - 120)$.

EXPLORE MORE

Q8 Because of the symmetry parabolas exhibit, the x-coordinate of the vertex is the average of the two roots, or $(r_1 + r_2)/2$. To find the y-coordinate of the vertex, substitute $(r_1 + r_2)/2$ for x in the original equation:

$$y = a\left(\frac{r_1 + r_2}{2} - r_1\right)\left(\frac{r_1 + r_2}{2} - r_2\right)$$

Simplifying, you get:

$$y = \frac{-a}{4}(r_1 - r_2)^2$$

Q9 The vertex will trace out a parabola whose vertex is at $(r_2, 0)$, opens in the opposite direction of the given parabola, and is the same shape. The equation is $y = -a(x - r_2)(x - r_2)$ or $y = -a(x - r_2)^2$.

WHOLE-CLASS PRESENTATION

Use **Factored Form Present.gsp** to present this activity to the whole class, following the directions and using the buttons in the sketch.

Parabolas in Standard Form

The vertex form of a parabola, $y = a(x - h)^2 + k$, and the factored form, $y = a(x - r_1)(x - r_2)$, both provide useful information about the graphs they describe. Perhaps the most common equation form for parabolas, though, is the standard form, $y = ax^2 + bx + c$. In some ways this is the simplest way to express a quadratic function, but it takes a little more effort to connect this form to the shape and position of the graph.

SKETCH AND INVESTIGATE

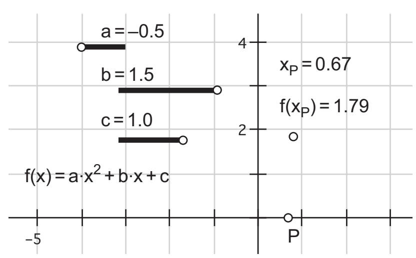
1. Open **Standard Form.gsp.** The sketch contains sliders labeled *a, b,* and *c.* You can change the values of the sliders by dragging the points at their tips.

First, create a function, evaluate it for an input variable, and plot the resulting point.

2. Use the slider values to define the function $f(x) = ax^2 + bx + c$. To do this, choose **Graph | New Function.** The New Function Calculator appears. If necessary, move it so that you can see the measurements of *a, b,* and *c.*

To enter *a, b,* and *c,* click their measurements in the sketch. To enter *x,* click the x key in the dialog box.

3. Enter $a*x^2 + b*x + c$ and click OK. Sketchpad creates the function.
4. Select *P* and measure its *x*-coordinate by choosing **Measure | Abscissa (x).**
5. Calculate $f(x_P)$, the value of function *f* evaluated at x_P. To do so, choose **Measure | Calculate.** Click the function object $f(x)$ and then measurement x_P. Finally, click OK. The value of the function appears.
6. To plot the point, select x_P and $f(x_P)$ and choose **Graph | Plot As (x, y).**



To turn tracing on or off, select the plotted point and choose **Display | Trace Plotted Point.**

Q1 Drag *P* back and forth along the *x*-axis to change the input variable for the function. How does the plotted point behave? Turn on tracing for the plotted point to better observe its behavior. What shape does this function trace out?

Traces are temporary in Sketchpad. Instead of using traces, you'll now make the graph permanent so you can compare the graphs of different functions.

7. Turn off tracing for the plotted point. Then choose **Display | Erase Traces** and drag *P* to make sure the traces no longer appear.

8. Plot a permanent graph of the function by selecting the function and choosing **Graph | Plot Function.** Drag *P* back and forth to make sure that the graph really corresponds to the path of the plotted point.

EXPLORING FAMILIES OF PARABOLAS

To make a line dashed, select it and choose **Display | Line Width | Dashed.**

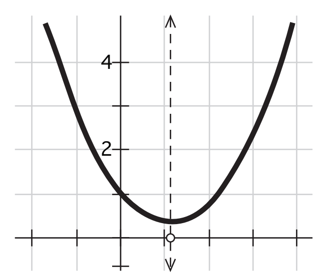
By dragging point *P*, you explored how x and $f(x)$ vary for one particular function with specific values of a, b, and c. Now you'll change the values of a, b, and c, changing the function itself and allowing you to explore whole families of parabolas.

Q2 Adjust slider a and describe its effect on the parabola. Discuss the effect of a's sign (whether it's positive or negative), its magnitude (how big or small it is), and anything else that seems important.

Q3 Dragging a appears to change all the points on the parabola but one: the y-intercept. Adjust each slider in turn and observe the effect on the y-intercept. How is the location of the y-intercept related to the values of the three sliders?

Q4 Adjust slider c. Describe how the parabola is transformed as c changes.

Q5 Adjust slider b. What happens to the parabola as b changes?



The *axis of symmetry* is the line that you can flip a parabola across without changing its position. In the next several steps you'll show that the axis of symmetry for any parabola in standard form is the vertical line that passes through the point $(-b/(2a), 0)$ on the x-axis.

To enter a and b, click on their measurements in the sketch.

9. For the x-coordinate of this point, use the Calculator to compute $-b/(2a)$.
10. For the y-coordinate, choose **Graph | New Parameter.** Name the new parameter *zero* and set its value to 0.
11. Select in order the $-b/(2a)$ calculation and the parameter *zero,* and choose **Graph | Plot As (x, y).** The axis of symmetry should pass through this point.
12. To actually construct the axis, select both the plotted point and the x-axis, and choose **Construct | Perpendicular Line.** Label this line *axis.*
13. Make line *axis* dashed, and mark it as a mirror by selecting it and choosing **Transform | Mark Mirror.**

To verify that this line really is the axis of symmetry, you can put a point on the graph and reflect the point across the line.

14. Use the **Point** tool to construct a point on the graph. Label it Q. Then with the point selected choose **Transform | Reflect.**

Q6 Drag point Q along the graph. What do you notice about the reflected point?

Q7 Change the shape of the parabola by dragging sliders a, b, and c. What happens to line *axis* as you change the parabola's shape? Drag point Q again. How does the plotted point behave?

Q8 Write the equations in standard form for three parabolas with different a values that have the same axis of symmetry. Check your work by adjusting the sliders in the sketch.

You can do some of these problems more easily by starting with the vertex form or factored form and converting to standard form when you're done.

Q9 Write the equation in standard form $ax^2 + bx + c$ for each of the parabolas described. As a check, adjust the sliders so that the parabola is drawn on the screen.

a. vertex at (2, 2); y-intercept at (0, 4)

b. vertex at (−2, 3); contains the point (0, 11)

c. x-intercepts at (−3, 0) and (5, 0); vertex at (1, −4)

d. contains the points (0, −4), (1, −1), and (2, 1)

EXPLORE MORE

Q10 The quadratic formula, which gives the roots of a parabola in terms of a, b, and c, is usually written

$$x_{\text{roots}} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By the rules of fraction addition, the right side can also be written as

$$\frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

You've already seen that the first part of this expression represents the x-value of the axis of symmetry. Show that the second part of the expression represents the distance from the axis of symmetry to each of the roots.

Objective: Students plot the graph of a quadratic function in standard form, investigate the relationship between this form and its graph, and use their observations to create functions from various descriptions of their graphs.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should have some experience with graphing quadratic equations.

Sketchpad Level: Intermediate. Students create and evaluate a function, plot and trace a point, and plot a function. They also create calculations and reflect a point across a mirror.

Activity Time: 35–45 minutes. The activity will take less time if students have already done an activity on the vertex form or the factored form. You can skip Q9 to save time.

Setting: Paired/Individual Activity (use **Standard Form.gsp**) or Whole-Class Presentation (use **Standard Form Present.gsp**)

Related Activities: Parabolas in Factored Form, Parabolas in Vertex Form, Changing Quadratic Function Forms

It's important that students understand that a parabola remains a parabola regardless of the form of its equation. Any parabola's equation can be written in many different forms. The particular forms we study are the ones that are especially simple or that convey special information.

SKETCH AND INVESTIGATE

Q1 As you drag *P*, the plotted point moves up or down, staying directly above or below *P*. This makes sense, because the point was constructed to have the same *x*-value as *P*. The plotted point follows the shape of a parabola. The precise shape depends on how the student has set the three parameters.

EXPLORING FAMILIES OF PARABOLAS

Q2 If *a* is positive, the parabola opens up; if *a* is negative, the parabola opens down. The larger the absolute value of *a*, the narrower the parabola. The closer *a* is to zero, the wider the parabola.

Q3 Neither parameter *a* nor parameter *b* has any effect on the *y*-intercept. The value of parameter *c* exactly matches the *y*-intercept.

Q4 Adjusting slider *c* moves the parabola up and down. The transformation is a vertical translation.

Q5 This is a translation. The shape and orientation of the parabola stay constant, but the axis of symmetry and both the *x*- and the *y*-coordinates of the vertex change. The path of the vertex is itself a parabola.

Q6 The reflection of point *Q* also remains on the graph, but on the opposite side.

Q7 As the parabola changes shape, the axis moves so that it always goes through the vertex, splitting the parabola into two halves that are mirror images. Dragging *Q* confirms that its reflection stays on the parabola no matter what the shape of the parabola is.

Q8 There are many possible answers. For example, each of the following has an axis of symmetry of $x = 2$: $y = 1x^2 - 4x + 13$, $y = -1x^2 + 4x - 10$, and $y = 3x^2 - 12x + 127$. (The value of *c* is irrelevant in every case—it doesn't affect the axis of symmetry.)

Q9
a. $y = 0.5x^2 - 2x + 4$
b. $y = 2x^2 + 8x + 11$
c. $y = 0.25x^2 - 0.5x - 3.75$
d. $y = -0.5x^2 + 3.5x - 4$

Make sure that students find the equations of these parabolas using paper-and-pencil calculations and use Sketchpad to check their answers.

EXPLORE MORE

Q10 One approach is to make measurements in the sketch and to verify the statement numerically. For instance, you can use the Calculator to calculate the *x*-coordinate of one of the *x*-intercepts using the quadratic formula.Use **Plot As (x, y)** to plot the *x*-intercept, using the zero parameter for its *y*-coordinate. Now measure the distance from the *x*-intercept to the axis of symmetry using **Measure | Coordinate Distance.** Compare the result with the result you get when calculating the value of the second part of the quadratic formula.

To derive the result algebraically, it's easier to use the factored form of the equation. See the activity Changing Quadratic Function Forms for some ideas on how to proceed.

Changing Quadratic Function Forms

Below are three widely used forms of a quadratic function.

Standard: $ax^2 + bx + c$ Vertex: $a(x - h)^2 + k$ Factored: $a(x - r_1)(x - r_2)$

Which is better? That really depends on the situation. What information do you have? What do you need? It can even depend on your own personal style. Whatever the case, it is useful to be able to switch from one form to the other.

VERTEX TO STANDARD

Converting a function from vertex form to standard form is really just a matter of expanding the vertex expression.

$$ax^2 + bx + c = a(x - h)^2 + k$$

$$ax^2 + bx + c = a(x^2 - 2hx + h)^2 + k$$

$$ax^2 + bx + c = ax^2 - 2ahx + ah^2 + k$$

As you may already know, the parameter *a* has the same effect on all three quadratic function forms. Perhaps that's why we use the same variable name in all three versions.

If both sides express the same function, corresponding coefficients must be equal.

$$a = a \qquad b = -2ah \qquad c = ah^2 + k$$

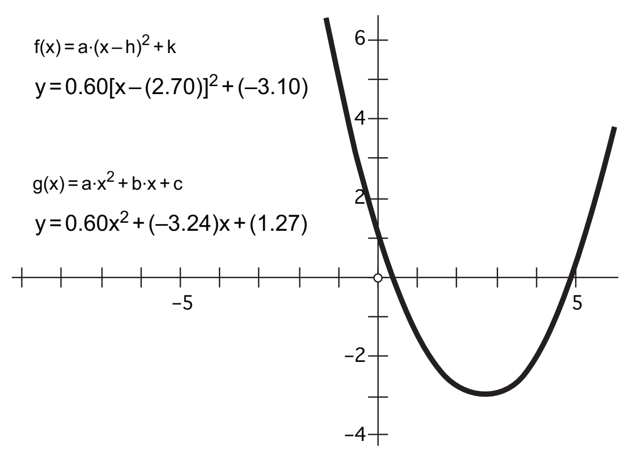
1. Open **Changing Forms.gsp.** The Vertex To Standard page already has a quadratic function in vertex form along with its graph. Change the parameters to verify that everything works as you would expect.

To use the Calculator, choose **Measure | Calculate.**

2. Use Sketchpad's Calculator to compute $-2ah$, and change the label of the result to b. Use the Calculator again to compute $ah^2 + k$, and change this label to c.

You will not be able to change the values of *b* or *c* directly. The parameters *a, h,* and *k* are still controlling everything.

3. Press the *Hide Vertex Form Graph* button. Choose **Graph | Plot New Function.** Enter the function in standard form: $g(x) = ax^2 + bx + c$.



By using the Hide/Show button, you can compare the two graphs to verify that the two function forms are equivalent. If the graphs do not overlay precisely, go back and check your calculations.

4. In this sketch you can find the red text "$=y = \{1\}x^2 + (\{2\})x + (\{3\})$". This is a template, which will help complete the presentation. Select in order the template and the calculations *a, b,* and *c*. Choose **Edit | Merge Text.**

Q1 What is $2(x - 8)^2 + 5$ in standard form?

STANDARD TO VERTEX

Converting standard form to vertex form is a bit more involved. Naturally, *a* is still *a*. One way to find *b* and *c* is to make use of these two equations from the previous section:

$$b = -2ah \qquad c = ah^2 + k$$

Use the first equation to write *h* in terms of *a* and *b*. Substitute the result into the second equation and write *k* in terms of *a, b,* and *c*.

Q2 What are *h* and *k* in terms of *a, b,* and *c*?

5. Open the Standard To Vertex page. Using the techniques from the previous section, construct a presentation for converting standard form to vertex form. Here is a summary of the steps:

- Create calculations for *h* and *k*.
- Create and plot a new function in vertex form using *a, h,* and *k*.
- Using the template and the calculations, merge the text and present the new function form.

Q3 Change parameter *a* to zero. The vertex form graph should disappear entirely. Does this mean that you can express certain quadratic functions in standard form but not vertex form? Explain.

OTHER CONVERSIONS

There are still four function conversions left. Open the document pages in order and use the summary in step 5 as a guide.

Factored to Standard

Moving from factored form to standard form works the same way as the conversion you did in the first section. Expand the expression.

Q4 How did you define *b* and *c*?

Standard to Factored

Remember that in factored form, $a(x - r_1)(x - r_2)$, the parameters r_1 and r_2 are the roots of the function. Use the quadratic formula to find the roots in terms of a, b, and c.

Q5 How did you define r_1 and r_2?

Q6 Under certain conditions, you cannot convert the function from standard form to factored form. What are those conditions? Explain.

Vertex to Factored

As in the previous case, the parameters r_1 and r_2 are the roots. You can find the roots by setting the function equal to zero and solving for x. Find the two solutions to this equation:

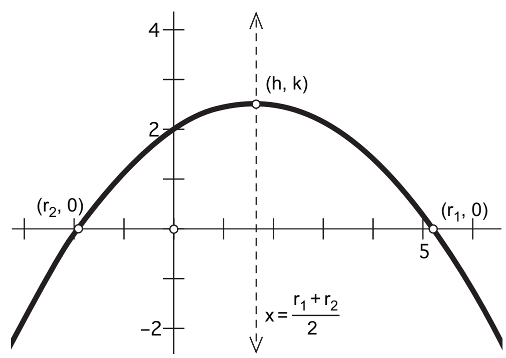
$$a(x - h)^2 + k = 0$$

Q7 How did you define r_1 and r_2?

Q8 Under what conditions is it not possible to convert a vertex form quadratic function into factored form?

Factored to Vertex

There is a geometric shortcut for this conversion. The coordinates of the vertex are (h, k) and the x-intercepts are r_1 and r_2. The vertex is on the axis of symmetry of the parabola. The x-intercept points are reflections of each other across the axis of symmetry. Therefore, the x-coordinate of the vertex must be the mean of the roots. That gives you h. The parameter k is $f(h)$.



Q9 How did you define h and k?

Objective: Students rewrite functions in order to change them between standard, vertex, and factored forms. They edit sketches so that the changes happen interactively.

Student Audience: Algebra 2/Precalculus

Prerequisites: This activity is quite challenging in terms of both algebra skills and Sketchpad skills. Students should be familiar with the properties of all three forms of quadratic functions, and they must be able to perform symbolic manipulations of algebraic expressions.

Sketchpad Level: Challenging. Students perform a number of calculations on the screen. They also plot functions and use prepared templates to merge text.

Activity Time: 50+ minutes. The activity is broken into six similar tasks. It has extensive guidance on the first, and guiding hints after that. The end of each task is a clean breaking point, so you could abbreviate the activity or perform it over more than one session.

Setting: Paired/Individual Activity (use **Changing Forms.gsp**)

A few students may get through this quickly, but most will need a lot of time. It will be especially difficult for students who have poor computer skills. With so many expressions to enter, they may tend to make mistakes with order of operations. It is important not to rush them. Encourage them to work together.

VERTEX TO STANDARD

Q1 $2x^2 - 32x + 133$

STANDARD TO VERTEX

Q2 $h = -\dfrac{b}{2a}$ $\qquad k = \dfrac{4ac - b^2}{4a}$

Q3 When $a = 0$, the formulas for both h and k are undefined. However, it is also true that when $a = 0$, the standard form function is linear, not quadratic. In fact, you can rewrite any quadratic function from standard form to vertex form.

OTHER CONVERSIONS

Factored to Standard

Q4 $b = -a(r_1 + r_2)$ $\qquad c = ar_1r_2$

Standard to Factored

Q5 $r_1 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$ $\qquad r_2 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$

Q6 The condition $a \neq 0$ occurs here once more, but again, that would only exclude functions that are not quadratic anyway. The only real problem is having a negative discriminant. You cannot use factored form if $b^2 - 4ac < 0$.

Vertex to Factored

Q7 $r_1 = h - \sqrt{-\dfrac{k}{a}}$ $\qquad r_2 = h + \sqrt{-\dfrac{k}{a}}$

Q8 In this case, either k and a must have opposite signs, or k must be zero. Otherwise, r_1 and r_2 would both be undefined. This happens only when the function has no real roots.

Factored to Vertex

Q9 $h = \dfrac{r_1 + r_2}{2}$ $\qquad k = -\dfrac{a(r_1 - r_2)^2}{4}$

Students may give the formula for k in various unsimplified forms. They may even answer simply $k = f(h)$. If they have calculated h, then it is possible to do that in Sketchpad.

The Discriminant

All quadratic functions have zero, one, or two real roots. The discriminant is a useful calculation that tells you how many roots there are for a given quadratic function.

This is the general definition of a quadratic function in standard form:

$$f(x) = ax^2 + bx + c, \text{ where } a \neq 0$$

If the function has roots, you can find them with these formulas:

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \qquad r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

The discriminant is the part under the radical signs. We often write it as Δ (delta).

$$\Delta = b^2 - 4ac$$

SKETCH AND INVESTIGATE

1. Open **Discriminant.gsp.** It has the graph of the quadratic function $f(x)$. Above the function definition are parameters a, b, and c.
2. Experiment by changing the parameters and observing the changes in the graph. When you finish, set the parameters so that the function now has this definition:

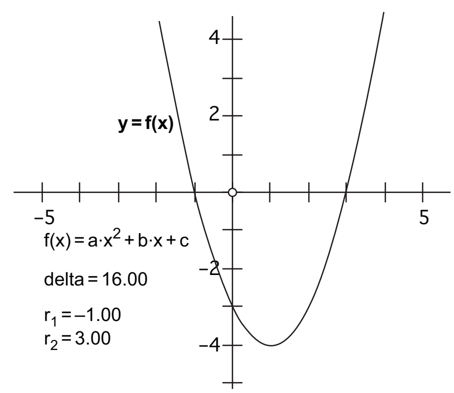
$$f(x) = x^2 - 2x - 3$$

To show the Calculator, choose **Measure | Calculate.**

To put a subscript at the end of a label, enclose the subscript in square brackets. For instance, r[1] will appear as r_1.

3. Use Sketchpad's Calculator to compute the two roots and the discriminant, using the formulas above and the parameters on the screen.
4. Label the calculations r_1, r_2, and *delta*.

Q1 What are the roots and the discriminant? Does the graph confirm the root calculations? Explain.



5. Select parameter c and tap the + key. Observe the changes. Stop when $c = 1$.

Q2 At this point, the two roots should be equal. Use the discriminant to explain why this is true. Describe the graph.

6. Increase parameter c further. Let $c = 2$.

Q3 What are the values of the discriminant and the roots? How does the discriminant explain the roots? Describe the graph.

GRAPH INTERSECTIONS

It is often possible to use the discriminant to find information about the intersections of two curves. The idea is to first work the equations into a single quadratic equation. Consider the graphs of these two general functions, one parabola and one line:

$$f(x) = a_1x^2 + b_1x + c_1 \qquad g(x) = mx + u$$

At any point where these graphs intersect, their y-coordinates must be equal. Therefore, you can set the functions equal to each other:

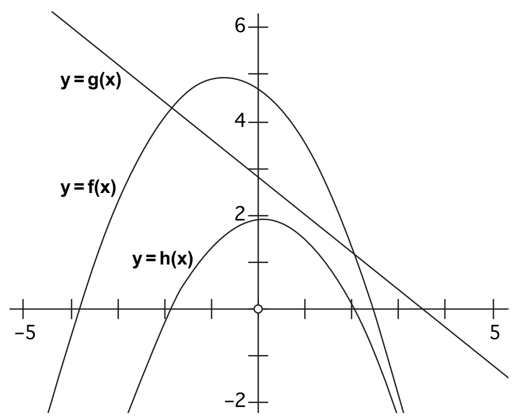
$$a_1x^2 + b_1x + c_1 = mx + u$$

Q4 Group the terms of this equation to form a quadratic function on the left side and zero on the right:

$$(\quad)x^2 + (\quad)x + (\quad) = 0$$

7. Go to page 2. Functions $f(x)$ and $g(x)$ are defined as shown above. Based on your answers to Q4, use the Calculator to compute values for a, b, and c.
8. Use the parameters from the previous step to calculate the discriminant.

Q5 Change the parameters of functions $f(x)$ and $g(x)$, and observe the graph. What do you see when the discriminant is positive? Zero? Negative?



9. Using the parameters from step 7, plot another quadratic function:

$$h(x) = ax^2 + bx + c$$

Q6 What is the relationship between the roots of function $h(x)$ and the intersection points of $f(x)$ and $g(x)$?

TANGENCY

10. Hide the graph of $h(x)$. Change the parameters so that function $f(x)$ has the definition below:

$$f(x) = -2x^2 - 3x + 2$$

Q7 What is the y-intercept of the graph of $f(x)$? Now change the parameters of $g(x)$ so that the two graphs have the same y-intercept and are tangent at that point. What is the new definition of $g(x)$? Explain how you got it.

Objective: Students calculate the discriminant of a quadratic function and observe its relationships to the roots of the function and to curve intersections and tangency.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students should be familiar with quadratic functions, their graphs, and the quadratic formula. Previous work with the discriminant is not necessary.

Sketchpad Level: Intermediate. Most of the work involves calculations and graphing.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Discriminant.gsp**) or Whole-Class Presentation (use **Discriminant Present.gsp**)

SKETCH AND INVESTIGATE

3. Students can simplify these calculations by finding delta first and then using that in the calculations for r_1 and r_2.

Q1 The roots are -1 and 3. The discriminant is 16. Since the roots are the solutions to the equation $f(x) = 0$, these are also the x-intercepts of the graph.

Q2 The only difference between the two root formulas is the sign of the radical. If the discriminant is zero, then the radical effectively goes away.

$$r_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = \frac{-b - \sqrt{0}}{2a} = -\frac{b}{2a}$$

$$r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-b + \sqrt{0}}{2a} = -\frac{b}{2a}$$

The graph is tangent to the x-axis at $(1, 0)$.

Q3 The discriminant is -4. Both of the roots are undefined. When the discriminant is negative, the root formulas include the square root of a negative number, which is why they are undefined. Since the function has no roots, its graph has no x-intercepts.

GRAPH INTERSECTIONS

Q4 $a_1x^2 + (b_1 - m)x + (c_1 - u) = 0$

7. Check in with students at this step. If they do not correctly define the parameters of the new function, nothing else in the activity will work out right.

$$a = a$$

$$b = b_1 - m$$

$$c = c_1 - u$$

Q5 When the discriminant is positive, the graphs intersect in exactly two points. When the discriminant is zero, the graphs are tangent. When the discriminant is negative, the graphs do not intersect.

Q6 The roots of $h(x)$ are the x-coordinates of the intersection points of the graphs of $f(x)$ and $g(x)$. Graphically, this means the intersection points are horizontally aligned with the x-intercepts of the graph of $h(x)$.

TANGENCY

Q7 The y-intercept of $f(x)$ is 2, the constant term of the function. In order to give $g(x)$ the same y-intercept, let $u = 2$. The graphs will be tangent if and only if the discriminant is equal to zero. Adjust the slope, m, until the discriminant is equal to zero.

$$g(x) = -3x + 2$$

Students may notice that these are the last two terms of $f(x)$. In fact, this works for any quadratic function. The last two terms define a linear function that shares the same y-intercept and is tangent at that point.

After finding that $u = 2$, students might find m analytically, by setting the discriminant equal to zero.

$$b^2 - 4ac = 0$$

$$(b_1 - m)^2 - 4a_1(c_1 - u) = 0$$

$$(-3 - m)^2 - 4(-2)(2 - 2) = 0$$

$$(-3 - m)^2 = 0$$

$$m = -3$$

The purpose of this presentation is to show the connections between quadratic functions and their discriminants. The discriminant can give quite a lot of information regarding roots, intersections, and tangency.

A SIMPLE EXAMPLE

1. Open **Discriminant Present.gsp.** Page 1 starts with a quadratic function written in standard form. Briefly review the definition.
2. Press the *Show Discriminant* button to show that definition, and press *Show Roots* to show where the discriminant appears in the quadratic formulas.

Q1 What happens if the discriminant is zero? (The roots are equal.) What happens if the discriminant is negative? (Both roots are undefined.)

3. Go to page 2. This is the graph of a quadratic function. The parameters a, b, and c control the coefficients of the function definition. Select parameter c and tap the + key.

Q2 As the graph moves upward, stop occasionally to ask the same questions. How many roots are there? Why? In each case, show how the number of roots is related to the sign of the discriminant. Give special attention to the tangency that occurs when the discriminant is zero.

4. Go to page 3. This shows two functions, one quadratic and one linear. Lead the class into a discussion of the intersection points of their graphs. For any given x, the functions represent the y-coordinates of the corresponding points on their graphs. At any intersection point, the graphs must have the same y-coordinate.

Q3 Press the $f(x) = g(x)$ button to set the functions equal. What do you get when you group like terms on the left side of the equation? (A single quadratic function is equal to zero.) Press *Group Terms* to do the same thing on the screen. Press *Show h(x)* and *Show Parameters* to identify the parameters of the new function.

5. Go to page 4. This is a graphic representation of the same problem, a quadratic $f(x)$ and a linear $g(x)$. Press *Show h(x)* to show the function that was derived on the previous page. Its discriminant will also appear. Change the function parameters.

Q4 What is the relationship between the discriminant and the intersections of $f(x)$ and $g(x)$? (The graphs have two intersection points when the discriminant is positive, are tangent when it is zero, and do not intersect when it is negative.)

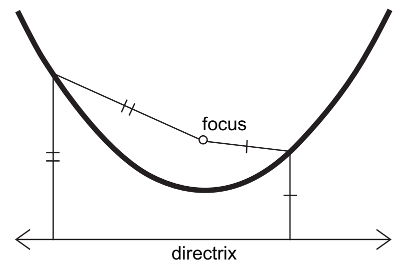
Q5 How do the intersections relate to the graph of $h(x)$? (They are horizontally aligned with the zeroes of $h(x)$.) Press *Show Intersections* to illustrate this fact.

Parabolas: A Geometric Approach

You may think of algebra and geometry as two very different branches of mathematics. In many ways they are. But you've seen that algebraic equations, such as $y = x + 3$, can be graphed as lines, which are geometric objects. Now you're studying parabolas—the graphs of equations such as $y = x^2 + 3$. Can parabolas be described geometrically, without using algebraic equations? In this activity you'll see that they can.

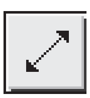
FOCUS AND DIRECTRIX

A *circle* can be described as the set of points in a plane that are the same distance from a fixed point—the center. Similarly, a *parabola* can be described as the set of points in a plane that are the same distance from a fixed point—the *focus*—as from a fixed line—the *directrix*.



This is a more complicated definition than the circle's, but as you do this activity it should make more and more sense.

A PARABOLA FROM SCRATCH

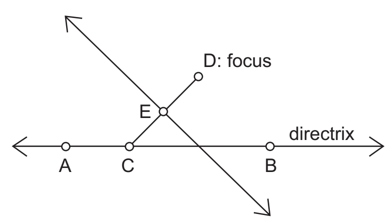


If the **Line** tool isn't showing, press down on the current **Straightedge** tool and choose the **Line** tool from the palette that pops up.

1. In a new sketch choose **Edit | Preferences.** On the Text panel, check Show Labels Automatically For All New Points. On the Color panel, check Fade Traces Over Time.
2. Using the **Line** tool, construct a horizontal line *AB*. (To make it exactly horizontal, hold the Shift key while you construct.) The line should be about a third of the way from the bottom of the sketch window. Show the line's label and change it to *directrix*.
3. Use the **Segment** tool to construct segment *CD* where *C* is on the line and *D* is about an inch above it.

Click on an object with the **Text** tool to show its label. Double-click the label itself to change it.

4. Change point *D*'s label to *D: Focus*.
5. Construct the midpoint of segment *CD* by selecting it and choosing **Construct | Midpoint.**
6. Select segment *CD* and midpoint *E*, and choose **Construct | Perpendicular Line.**



After step 6

The line you just constructed—perpendicular to segment *CD* through its midpoint—is called the *perpendicular bisector* of segment *CD*.

Q1 Imagine a point anywhere on the perpendicular bisector. How do you think the point's distance to *C* compares to its distance to *D*?

7. Construct a line perpendicular to $\overleftrightarrow{AB}$ and passing through point *C*.

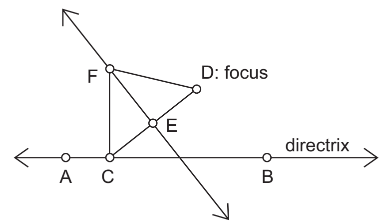
8. Select the two perpendicular lines (the ones constructed in the last two steps) and choose **Construct | Intersection.**

You just constructed point *F*—the first point on the parabola.

To hide objects, select them and choose **Display | Hide.**

9. Hide $\overleftrightarrow{FC}$. Then, using the **Segment** tool, construct $\overline{FC}$ and $\overline{FD}$.

10. Measure the lengths of the two new segments. To do this, select them and choose **Measure | Length.**



After step 9

Q2 Using the **Arrow** tool, drag *C* back and forth along line *AB*.

Refer back to the parabola definition in the introduction.

What do you notice about the length measurements as you drag *C*? Explain how this demonstrates that *F* is indeed always a point on the parabola defined by the given focus and directrix.

11. Select point *F* and choose **Display | Trace Intersection.** Now once again drag *C* back and forth. Point *F* will leave a trail as it traces out the parabola.

12. Drag the focus away from the directrix. Again, drag point *C*. Notice how this curve compares to the previous one. Now drag the focus closer to the directrix than it was originally and repeat the process.

Tracing the curve in this way works well enough to show you the shape of the parabola. But it can get a little annoying having to drag point *C* again and again. Here's a more efficient approach.

Select the point and choose **Display | Trace Intersection** to turn tracing on or off.

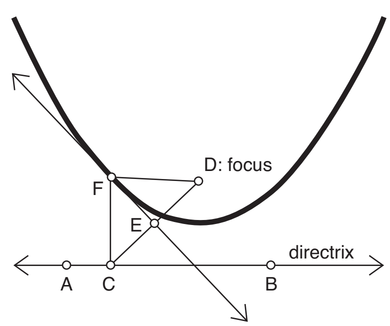
13. Turn off tracing for point *F*.

14. Select points *F* and *C*, and choose **Construct | Locus.**

    The entire curve appears. It's called the *locus* of point *F* as point *C* moves along $\overleftrightarrow{AB}$.

To hide objects, select them and choose **Display | Hide.**

15. To make the diagram less cluttered, hide everything except the parabola, the focus, and the directrix.



16. Drag the focus around and observe how the curve changes.

Q3 What happens to the parabola as the focus is dragged farther away from the directrix? Closer to it?

Q4 What happens to the parabola when the focus is dragged below the directrix?

17. Save your sketch. You may want to use it for another activity, Building Headlights and Satellite Dishes.

EXPLORE MORE

Q5 Use the commands in the Construct menu to construct the vertex of your parabola. This point should remain the vertex no matter where you drag the focus and the directrix.

Q6 Show that the equation of a parabola with its vertex at the origin and its focus at $(0, p)$ is $y = \frac{x^2}{4p}$. To do this, select the vertex you constructed in Q5 and choose **Define Origin** from the Graph menu. Then measure the y-coordinate (ordinate) of the focus and relabel this measurement p. Now use **Graph | Plot New Function** to plot $\frac{x^2}{4p}$. This plot should be right on top of the parabola, even if you drag the focus or the directrix.

Objective: Students use a locus definition to construct a parabola geometrically and relate the result to parabolas that are plotted algebraically.

Student Audience: Algebra 1/Algebra 2

Prerequisites: This activity works best if students have already been learning about parabolas from an algebraic perspective, but it also works as a first introduction to parabolas.

Sketchpad Level: Intermediate

Activity Time: 30–45 minutes depending on students' Sketchpad proficiency

Setting: Paired/Individual Activity (The document **Parabolas Geometric.gsp** shows the construction at several stages.)

Related Sketch: Conic Sections.gsp allows you to vary the angle at which a plane intersects a cone and to view the intersection from various directions.

A good way to introduce this activity might be to talk a little about the history of parabolas. Parabolas have been studied for over 2000 years, but algebraic equations for them have only been known about for the past 500 years or so. How did people describe and define parabolas before? How were they drawn without coordinate geometry? Where do parabolas come from? (They are sections of cones created by a cutting plane parallel to the side of a cone). This kind of historical perspective may help students appreciate the true power of the algebraic equations.

CONSTRUCTION TIPS

2. Holding down the Shift key while drawing the line helps keep it horizontal.
3. Perform your first click on the directrix itself (though not on one of its control points). (The directrix should be highlighted before you click.)
5. Use the **Arrow** tool in this step.
7. Refer to step 6 for a reminder how to do this.
11. Make sure only point *F* is selected here. To deselect all objects, click in blank space.
12. Traces will gradually fade because of the Preference setting made in step 1. If you prefer that traces not fade, uncheck Fade Traces Over Time on the Color panel of the Preferences dialog box. To clear traces from the screen, choose **Display | Erase Traces.**
13. Again, make sure only *F* is selected.

A PARABOLA FROM SCRATCH

Q1 It's the same distance from either endpoint. This is a theorem from geometry and can be proven, but for our purposes it's probably okay just to leave this up to students' common sense.

Q2 The two length measurements are always equal to each other. (This should confirm the students' conjectures from Q1.) The length of segment *FC* is the distance from *F* to the directrix, and the length of *FD* is the distance from *F* to the focus. Point *F*, then, fits the definition from the activity's introduction and is therefore on the parabola defined by the given focus and directrix.

Q3 The farther point *C* is from the directrix, the wider the parabola; the closer, the skinnier.

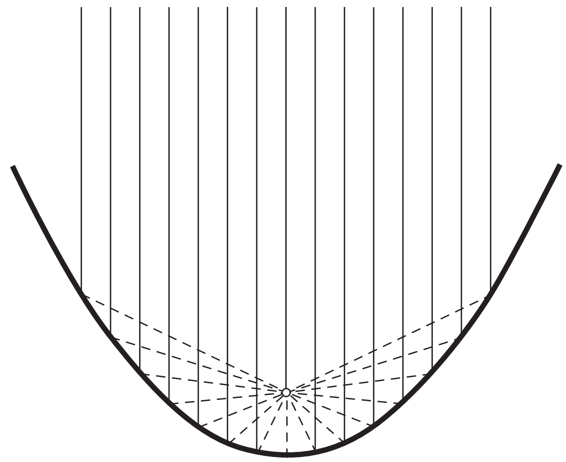
Q4 When point *C* is below the directrix, the parabola opens downward.

EXPLORE MORE

Q5 Select the focus and the directrix, and choose **Construct | Perpendicular Line.** Click on the spot where this new line intersects the directrix to construct the point of intersection there. Hide the line and then construct a segment between the focus and the new point. Construct the midpoint of this segment using **Construct | Midpoint.** This point is the vertex of the parabola.

Parabolas in Headlights and Satellite Dishes

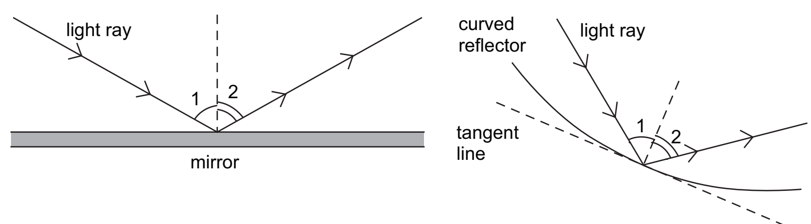
Have you ever wondered why satellite dishes are shaped the way they are? And what is it about the way headlights are designed that makes the light travel outward in one direction? It turns out that both devices use *parabolic reflectors* because of their special reflective properties. In this activity you'll construct a two-dimensional model of a parabolic reflector and explore what makes it ideal for reflecting and collecting rays.



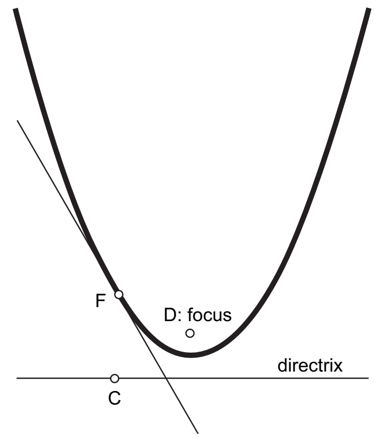
BACKGROUND

The two angles are measured relative to the *normal* line—the line that's perpendicular to the tangent.

In order for this activity to make sense, there are two things you need to know about the *Law of Reflection*. First, when a ray of light reflects from a flat surface, such as a mirror, the *angle of incidence* ($\angle 1$ below) equals the *angle of reflection* ($\angle 2$ below). This is a fancy-sounding law of physics that states something very simple: Light bounces off at the same angle it hits. Second, if light bounces off a *curved* surface, the Law of Reflection still holds—just imagine the light bouncing off a line *tangent* to the curve.



SKETCH AND INVESTIGATE



1. If you have completed the activity Parabolas: A Geometric Approach, open your sketch from that activity. Otherwise, open **Reflector.gsp** and skip to step 4.

To hide objects, select them and choose **Display | Hide.**

2. Choose **Display | Show All Hidden.** Then hide all but the following objects: the parabola, focus, directrix, points *C* and *F,* and the line that's tangent to the parabola.
3. To make the parabola flatter, drag the directrix down near the bottom of the window and the focus about two-thirds of the way to the top.

You need a light source—the tiny bulb inside a headlight whose light bounces off the reflector into the outside world.

4. Using the **Point** tool, draw a new point *G* inside the parabola somewhere not too far from the focus.

Now you need a ray of light coming from the source and hitting the reflector.

Either use the **Segment** tool or select the two points and choose **Construct | Segment.**

5. Construct a segment between points *G* and *F.*

The bulb actually sends out rays of light in all directions. Use a Sketchpad locus construction to show 15 of these rays.

6. Select the new segment and point *C,* and choose **Construct | Locus.** With the locus selected, choose **Display | Line Width | Dashed.** With the locus still selected, choose **Edit | Properties.** On the Plot panel, set the number of samples to 15.

You'll see 15 rays coming from the light source and hitting the reflector.

The normal line is perpendicular to the tangent at the point where the ray hits the reflector.

Each ray must bounce off the reflector at the same angle at which it hit the reflector. The easiest way to construct this involves the *normal* line.

7. To construct the normal line, select point *F* and the line that's tangent to the parabola. Then choose **Construct | Perpendicular Line.**

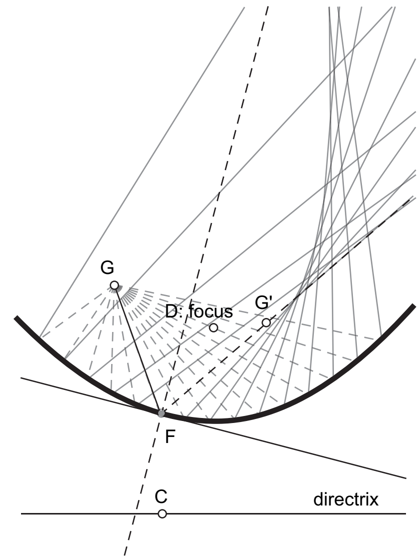
The symmetry of reflection guarantees that the incoming ray and the outgoing ray will both make the same angle with the normal.

8. Double-click the new line to mark it as a mirror for reflection. Now select point *G* and choose **Transform | Reflect.** The reflected image of point *G*, G', appears.
9. Select in order point *F* and point G'. Then choose **Construct | Ray.**

This ray is the original ray of light after it's bounced off the reflector. By reflecting *G* across the perpendicular line, you guaranteed that the angle of incidence would equal the angle of reflection, as the Law of Reflection states.

10. Repeat step 6 using the new ray in place of the segment. Give the new locus a thin line weight (instead of dashed) to distinguish it from the other, and set the number of samples to 15 as well.

You should now see the 15 rays of light coming from the light source and bouncing off the reflector.



After step 10

Q1 Drag the light source around and observe how its location affects the reflected light. Where does the light source need to be in order for the light to travel out in parallel rays, as with car headlights?

11. Merge the light source and the focus. To do this, select these two points and choose **Edit | Merge Points.**

The two points merge into one. Now the light source is at the focus and light is bouncing off in parallel rays.

Q2 Drag the focus and the directrix separately to change the shape of the reflector. Do the reflected rays remain parallel?

12. Clean up the sketch by hiding segment *GF,* the perpendicular line through point *F,* the ray originating at *F,* point *G′* defining that ray, and anything else you'd like out of the way.

Q3 A satellite dish operates similarly to a car's headlights, except that instead of emitting rays, it collects them. Explain how you would design a satellite dish.

EXPLORE MORE

Q4 Do some research to find other objects shaped like a parabolic dish. Explain why each of them uses this shape.

Parabolas in Headlights and Satellite Dishes Activity Notes

Objective: Students construct a two-dimensional model of a parabolic reflector and explore what makes it ideal for reflecting and collecting rays.

Student Audience: Algebra 1/Algebra 2

Prerequisites: This activity can stand on its own, or you can use it as a follow-up to the activity Parabolas: A Geometric Approach.

Sketchpad Level: Challenging. There are quite a few construction steps.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Reflector.gsp** if students don't have sketches from the activity Parabolas: A Geometric Approach)

Related Activities: Parabolas: A Geometric Approach, Conic Reflections

One way to introduce this activity is to ask students what shape a "burning mirror"—a mirror designed to reflect the sun's rays to one point—should be. Why would a straight line (flat mirror) be a bad choice? Students may be interested to know that Diocles discovered—and used—the answer to this question c. 200 B.C.E. Diocles was improving on work done a century earlier by Archimedes, who, as legend has it, used burning mirrors to set the Roman fleet ablaze as it entered Syracuse harbor. Diocles' principle is still used today in solar-powered steam generators. (See Toomer, G.J., *Diocles on Burning Mirrors.* New York: Springer-Verlag, 1976. The Archimedes legend was the subject of Mythbusters Episode #46 on the Discovery Channel.)

SKETCH AND INVESTIGATE

Q1 When the light source is at the focus, the rays will reflect parallel to each other.

Q2 Provided the light source is at the focus, the rays will reflect parallel to each other (and to the axis of symmetry of the parabola) no matter how narrow or wide the parabola is.

Q3 Place the sensor at the focus of the three-dimensional parabola (called a *parabaloid*). The signals coming in roughly parallel to the direction the dish is facing will reflect right to the sensor whereas all other signals will bounce and miss the sensor.

EXPLORE MORE

Q4 Another example of a parabolic dish is the parabolic microphones often seen on the sidelines of football games and used to catch the sounds of the game. These work the same way a satellite dish works: Sound waves coming in parallel to the direction the device is aimed bounce toward the microphone, which is located at the focus. Sound waves from other directions bounce harmlessly away.

Conic Reflections

You may already know that light rays from the focus of a parabola reflect to form parallel rays. (For instance, the reflector in a searchlight or a car's headlight is shaped like a parabola.) A parabola is the graph of a quadratic function, but it is also a conic section. As it turns out, all of the conic sections have special reflective properties.

REFLECTIONS IN AN ELLIPSE

Light and sound are very different in nature but share many common properties. You can imagine either of them as waves or rays.

Kidney stones are a very painful ailment caused by crystallization of salts inside the kidney. They are in fact small stones. A modern treatment is extracorporeal shock wave lithotripsy (ESWL). Patients are happy about the extracorporeal part, because the word means "outside the body." Surgery usually is not necessary. The idea is to create a sonic pulse outside so that the shock waves converge at the stone and break it into smaller pieces that can pass out of the kidney.

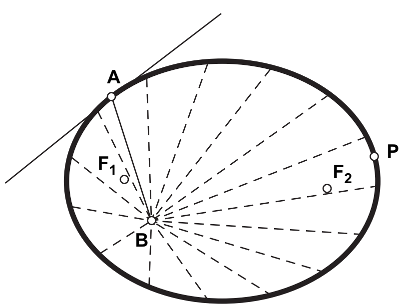
1. Open the Ellipse page of **Conic Reflections.gsp.** Drag the points to see how they affect the image. This particular ellipse is defined by the two foci (F_1 and F_2) and a free point *P* on the curve. Point *A* is attached to the ellipse, and the line through *A* is tangent to the ellipse.

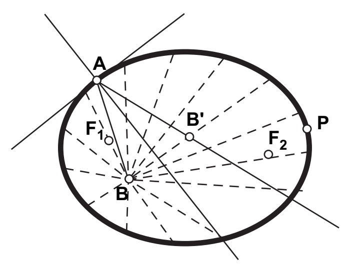
Point *B* represents the source of the waves of sound or light.

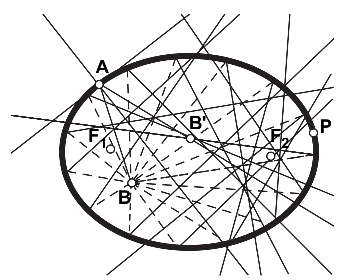
2. Construct point *B* in the interior of the ellipse. Construct line segment *BA*. This represents a ray of sound traveling from *B* and striking the surface at point *A*.

The locus shows the position of segment *BA* for various possible positions of point *A* on the ellipse.

3. Select point *A* and segment *BA*. Choose **Construct | Locus.** Select the locus and choose **Edit | Properties.** On the Plot panel, change the number of samples to 15. Select the locus again and change the line width to dashed. You should now see 15 rays coming from *B* and striking the surface at various points.







When a ray reflects from a surface, the angle of incidence is equal to the angle of reflection, and you measure those angles from a line that is perpendicular to a tangent at that point on the surface. This principle applies to all curved surfaces.

The normal line intersects the tangent, and the curve itself, at a right angle.

4. Select point *A* and the tangent line. Choose **Construct | Perpendicular Line.** This is called a *normal* line.

5. Double-click the normal line to mark it as a reflection line. Select point *B* and choose **Transform | Reflect.** Construct ray AB'. This is the reflected ray.

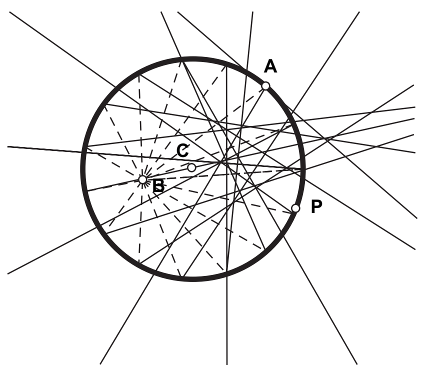
6. Construct the locus of the reflected ray. Again, change the number of locus samples to 15.

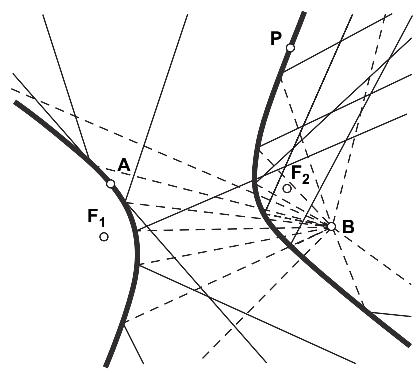
Q1 Point *B* is the source of the sonic pulses. Drag it around the screen and watch the pattern change. Where can you place it in order to get the reflected rays to converge at a single point? When this happens, where do the rays converge again?

Q2 You may notice that the rays come together at a point inside the ellipse. Why is this a problem for the ESWL treatment? Suggest a solution.

REFLECTIONS IN OTHER CONICS

There are still two conic sections—the circle and the hyperbola—that you have not investigated for reflective properties. The Sketchpad document contains a page for each of them. Repeat these same construction steps to explore their properties.





Q3 You can think of a circle as a special case of an ellipse. The two foci are in the same place, which is the circle center. With this in mind, where would you expect the ray reflections to go if their source is at the center of a circle?

Q4 The hyperbola may be a bit more difficult to see. Instead of watching where the rays go, try to see where they appear to be coming from. If the source of the rays is on one focus, where do the reflected rays appear to come from?

PRESENT

The Parabola page is already completed. It summarizes the findings from the Building Headlights and Satellite Dishes activity. On each of the other pages, use the **Text** tool to describe the special reflective properties of the curve.

Notice that the Parabola page has a button that moves point *B* to the focus. To create a similar button on the other pages, select in order point *B* and the destination point. Choose **Edit | Action Buttons | Movement.**

Objective: The reflective properties of parabolas are well known. In this activity students explore interesting reflective properties of other conic sections.

Student Audience: Algebra 2/Precalculus/Geometry

Prerequisites: Students should have introductory knowledge of the names and geometric properties of conic sections. They do not need any experience with the analytical representations.

Sketchpad Level: Challenging. The activity involves several locus constructions and editing of properties.

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Conic Reflections.gsp**) or Whole-Class Presentation (use **Conic Reflections Present.gsp**)

Related Activities: This activity is intended as an extension of the Parabolas in Headlights and Satellite Dishes activity. If the students have not performed that task, you should first spend some time explaining the properties shown in the Parabola page of the document. That sketch is completed, and it will give the class a better understanding of the objective.

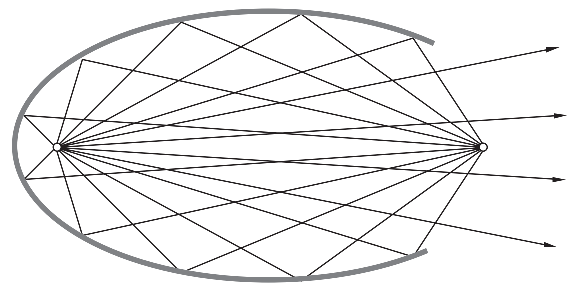
When you review this activity with your class, ask the students to explain the connection between conic sections and kidney stones.

REFLECTIONS IN AN ELLIPSE

Q1 When you drag point *B* to one focus, the rays are concurrent at the other focus.

Q2 Remember that the rays must converge inside a person's body. If that point is inside the ellipse, then the patient must be inside. Students may guess (incorrectly) that the reflecting chamber must be big enough to hold the patient.

In practice, the chamber is nowhere near that large. It is ellipsoidal in shape, but one end is open (see below). This still allows most of the shock energy to reach the target, but some of it passes out the open end.



REFLECTIONS IN OTHER CONICS

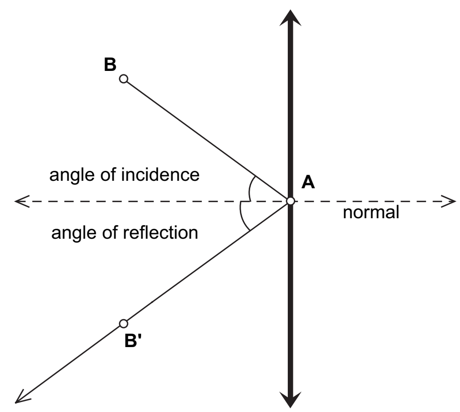
Q3 A circle is an ellipse with both foci at the same center point. Therefore, rays sent from the center reflect back to the center.

Q4 In a hyperbola, send the rays from one focus. After reflection, they will travel directly away from the other focus.

To present the reflective properties of conic sections, you can use the prepared document **Conic Reflections Present.gsp.** This document allows you to skip the construction work and present a summary of the observations. In general, just walk through the pages in order, pressing the action buttons in order from the top down.

1. Open the Flat Surface page of **Conic Reflections Present.gsp.** This shows a reflection on a flat surface, like a mirror or a pool table cushion. Point *B* is the source of a ray that strikes the surface at *A*.

2. Press *Show Normal Line*. Explain the physics principle that the angle of incidence is equal to the angle of reflection, and you measure both angles with respect to the normal line.



3. Press *Show Reflection*. Identify the angles of incidence and reflection. Drag points *A* and *B* to show the changes in the reflection.

Q1 Open the Curved Surface page. What happens when we have a curved surface? How do we find the normal line? (The normal is perpendicular to the tangent at the same point *A*.)

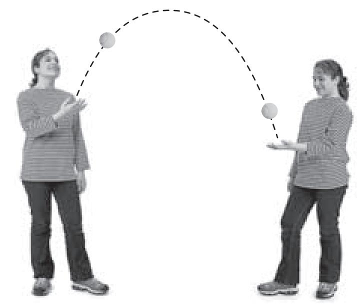
4. Show the hidden objects. When dragging point *B*, be sure to show that the same principle holds when it is on the concave side of the curve.

Q2 The remaining pages show reflective properties of conic sections in turn. Each page opens with a single reflected ray. Press *Show Rays* to show 15 rays all originating at point *B*. No particular pattern is visible until you drag point *B* to a focus. In each case, challenge students to predict what the reflected ray pattern will be. What will happen when we drag this to the focus? What practical applications can we find for this effect?

Q3 After showing that an ellipse reflects a ray from one focus to the other, drag point *B* away from the focus and hide the rays. Press *Move F_2 to F_1* and let students see that a circle is actually an ellipse with coincident foci. With this fact in mind, what should happen when point *B* is at the center of a circle? (The reflected rays go straight back to the center.)

5. The hyperbola reflection pattern may be more difficult to see, so give students more time. When point *B* is on F_1, the reflected rays move out away from the second focus rather than through it. Press *Show Extensions* to help them see the alignment.

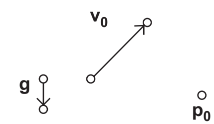
Modeling Projectile Motion



When you toss a ball into the air, the laws of physics control its motion through the air and back to the ground. In this activity you'll use Sketchpad's **Iterate** command to model a projectile as its position, velocity, and gravity interact.

To create the model, you'll make a sketch showing what happens to the projectile in a short period of time. You'll use iteration to repeat the construction and create a graph of the projectile's motion over a much longer period of time.

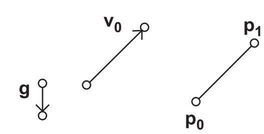
SKETCH



1. Open **Projectile Motion.gsp.** Notice the vector for gravity (g), the vector for the initial velocity (v_0) of the object, and the point for the initial position (p_0) of the object. Vector g points down to represent the downward acceleration gravity exerts on projectiles. Vector v_0 points diagonally to indicate the movement of the ball at the instant it is thrown.

At this instant, the ball has not moved from its initial position. To begin, you'll construct a segment to represent the ball's motion by using the velocity vector to find the ball's position after the first interval of time.

In the Translate dialog box, select Marked for all translations.

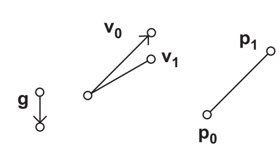


2. To mark the velocity vector as the vector for translation, select its initial and final points, and choose **Transform | Mark Vector.** To translate the position, select point p_0 and choose **Transform | Translate.** Label the translated position p_1 and connect the initial position and the translated position with a segment.

The new segment shows how the ball moves through the air during the first period of time. But if you've ever thrown a ball, you know that gravity affects its flight. In your sketch you must also take into account gravity's effect on the ball's velocity. To do this, you'll use the gravity vector to construct a new velocity vector.

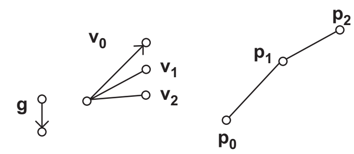
This segment represents the velocity vector at the end of the first period of time.

3. Mark the gravity vector as the vector for translation. Translate point v_0 by this vector, and then construct a segment from the original tail of the velocity vector to v_1 (the new translated point).



The v_1 vector represents the object's velocity at the end of the first interval of time. So v_1 describes the ball's velocity when it's at p_1.

4. Repeat steps 2 and 3 for the second period of time. Be sure you construct the new position segment from the end of the old one, but the new velocity vector with its tail in the same place as the existing velocity vectors.



Q1 How did the velocity change as the ball moved from p_0 to p_1? Is the ball going faster or slower? How did the direction of its motion change?

5. Repeat steps 2 and 3 for a third period of time.

Q2 What do you notice when you compare the velocities v_0, v_1, v_2, and v_3 for the three periods? How is the velocity vector changing the object's path?

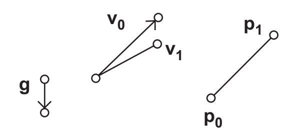
Q3 What do you notice when you compare positions p_0, p_1, p_2, and p_3? Describe the motion of the object.

Q4 Make a prediction for how the position will change during the next ten periods of time.

It would be a lot of work to do the same construction ten more times. You can use iteration to make the process easier.

Choose **Edit | Undo** repeatedly until you get back to the end of the first period.

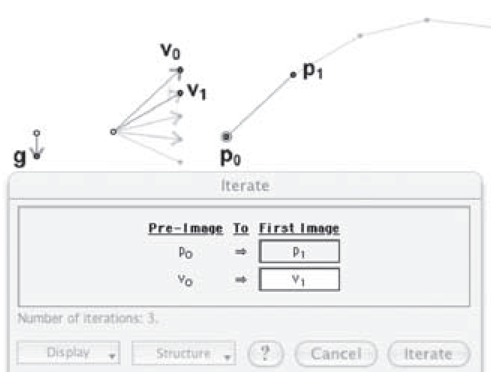
6. Undo your work back to the end of the first period of time.



Gravity is constant, so you didn't translate that vector. This means that you only have to keep track of changes in the position and the velocity. The first step of the iteration will map $p_0 \Rightarrow p_1$ and $v_0 \Rightarrow v_1$.

Points p_0 and v_0 are called the *pre-image* points of the iteration, and points p_1 and v_1 are the *image* points.

7. To create the iteration, select the pre-image points p_0 and v_0 and choose **Transform | Iterate.** Then click p_1 and v_1 in the sketch so that each pre-image point is correctly mapped to its image point. Click Iterate to complete the iteration and show the next three steps.



8. With the iterated images still selected, press the + key on your keyboard to increase the number of iterations until you have about 15 of them.

Q5 Describe the shape of the object's flight. For each period of time, how do the direction of the velocity vector and the path of the object relate to each other?

Q6 During which period of time does the velocity begin to point downward? Can you tell at what moment the ball begins to fall downward by looking only at the velocity vector? Can you tell when the velocity vector begins to point downward just by looking at your construction of the ball's path?

To turn gravity off and check your prediction, drag the end points of the gravity vector so that they coincide.

Q7 What do you think the path would look like if there were no gravity? Draw your prediction on paper.

Q8 How well does the shape of the path represent the actual flight of a ball through the air? Think of a time when you've seen a ball fly through the air. How does your constructed path differ from your observation of how balls fly through the air?

The model can be more accurate if you make each iteration represent a shorter period of time. To make this change, you will have to change the lengths of the original vectors.

Q9 To make each period of time half of its original value, how much shorter must you make the vectors? How many more iterations will you need to model the same period of time?

9. Shorten the gravity and initial velocity vectors and increase the number of iterations until your model shows smooth motion for the projectile.

Q10 What shape does the projectile's motion appear to have?

EXPLORE MORE

10. Using **Graph | Plot New** Function and the form $f(x) = ax^2 + bx + c$, try to match a parabola to the path of the ball. Adjust the parameters to change the shape of your parabola.

11. Extend the model to use a parameter to represent the time interval. (*Hint:* Use the parameter to dilate the original vectors before you iterate them.)

Objective: Students make a Sketchpad model of a physical process operating for a short period of time, and then iterate that process to see how it produces reasonable long-term behavior.

Student Audience: Algebra 1/Algebra 2/Physics

Prerequisites: It helps if students have seen a velocity vector used to show both how fast and in what direction an object is moving.

Sketchpad Level: Intermediate

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity (use **Projectile Motion.gsp**) or Whole-Class Presentation (use **Projectile Motion Present.gsp**)

SKETCH

Q1 During the first period of time, the velocity's magnitude is reduced and its direction is less steeply upward. Students may not be familiar with the terms *magnitude* and *direction*. It's not important that they learn the terms at this time, but it is important that they recognize both changes: The ball is going more slowly, and its direction has changed so that its path is less steep than it was at first. Students can see both effects by comparing vectors v_0 and v_1. Vector v_1 is shorter in length and is pointed less steeply upward.

Q2 With each interval of time, the velocity vector points more downward. As the velocity vector moves downward, the ball is not moving upward as steeply as it was at first (and it soon starts to move downward).

Q3 The path of the ball levels off, becoming more horizontal and less vertical. Soon the ball begins to move downward. (In the default state of the sketch, the ball reaches its apex near p_3 and then begins to move downward.)

Q4 If the velocity pattern continues, the ball's direction of travel will be more and more downward at each time interval. This agrees with students' knowledge of how balls fly through the air, but it is also important that they observe how the changes in the velocity vector correspond to the actual movement of the ball.

Q5 The shape of the ball's path looks like an arc of a parabola. The more downward the velocity vector points, and the longer it gets in this direction, the more vertical the ball's path becomes.

Q6 In the original state of the sketch, the ball begins to move downward during the fourth interval. The v_3 vector is the first one to point downward, with the result that after p_3, the ball begins to move downward.

Q7 Answers will vary. If there is no gravity, the velocity vector will not change over time. With no changes in velocity, the ball will continue to travel in the initial direction, and its path will be a straight line. Encourage students to explain their drawings, and make sure they understand that the ball moves in a straight line forever because gravity does not alter the ball's velocity.

Q8 Students have a mental image of how a thrown ball moves through the air, but it's still useful to demonstrate by gently tossing a soft object. The real ball moves through the air in a smooth arc, while the construction is a rough approximation because its position is only measured at intervals, where the segments connect.

Some students may raise a more sophisticated question about the terminal velocity of a ball and point out that the Sketchpad model doesn't allow for air resistance.

Q9 You should shorten both the velocity and gravity vectors by half. You should shorten the velocity vector because the ball should travel less far in a shorter interval of time. Similarly, you should shorten the gravity vector because during a half-second interval the ball's velocity should not change by as much as it does during a full-second interval.

Q10 As it gets smoother, the iterated path becomes more and more parabolic in shape.

EXPLORE MORE

10. Students can also make the parabola and the iterated path match by changing the path. This approach requires modifying the initial position of the ball, the initial velocity of the ball, and the length of the gravity vector. (The gravity vector must remain vertical or the objects won't match.)