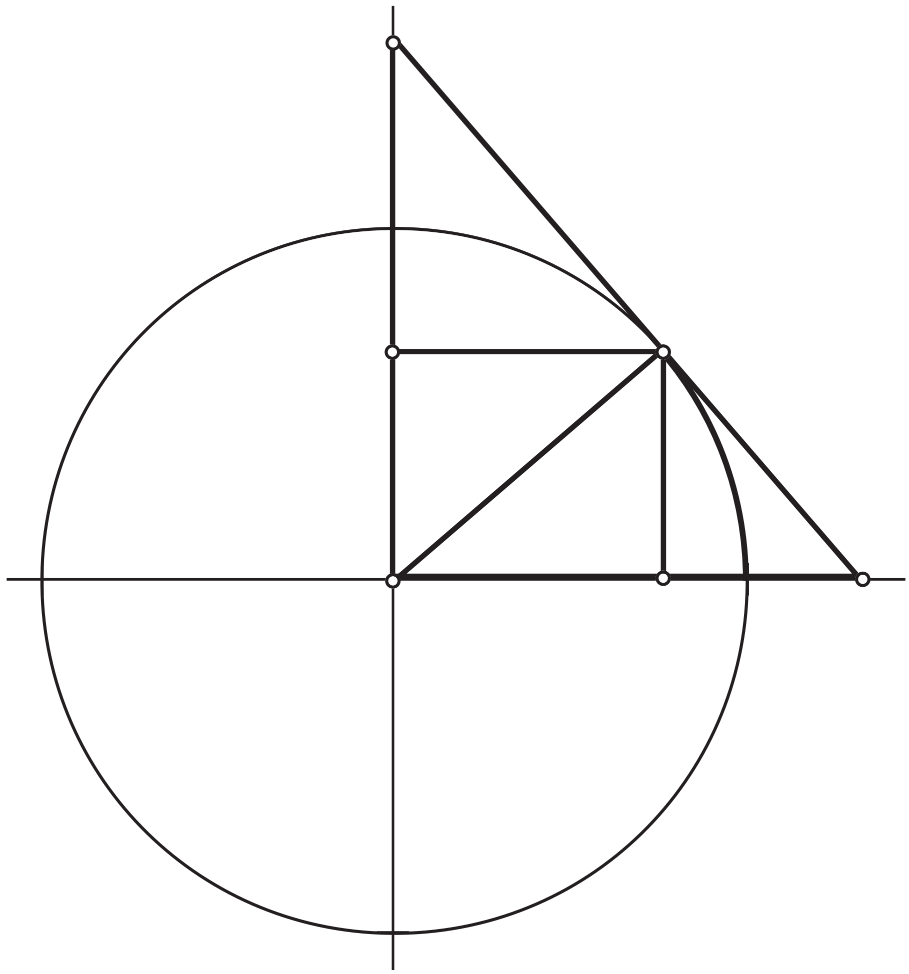
7

Trigonometric Functions



Right Triangle Functions

Within a right triangle there are many important and useful ratios. You may have used *SOH CAH TOA* as a way of remembering the names of certain right triangle ratios:

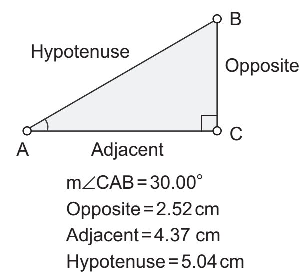
SOH: The ratio for *Sine* is *Opposite* over *Hypotenuse*.

CAH: The ratio for *Cosine* is *Adjacent* over *Hypotenuse*.

TOA: The ratio for *Tangent* is *Opposite* over *Adjacent*.

In this activity you'll explore these ratios in triangles of different shapes and sizes.

RATIOS



1. Open **Right Triangle Functions.gsp.** Drag each point and observe its effect on the size and shape of the triangle.
2. Use the **Text** tool to label the sides of the triangle *Opposite*, *Adjacent*, and *Hypotenuse* based on their position relative to $\angle CAB$.
3. Measure $\angle CAB$ and the length of each side.

To calculate the ratio, choose **Measure | Calculate** and enter each side into the calculation by clicking its measurement in the sketch.

Double-click the **Text** tool on the resulting calculation to change its label.

4. Calculate the sine ratio for $\angle CAB$ by dividing the appropriate sides of the triangle. Label the resulting ratio *sin A*.
5. Drag *B* and observe its effect on $\sin A$. Use the button to make $m\angle CAB = 30°$.

Q1 What is $\sin 30°$?

Q2 If you make the triangle larger without changing $m\angle CAB$, do you think $\sin A$ will increase, decrease, or stay the same? Give a reason for your prediction.

Q3 Change the size of the triangle by dragging the *Length of Hypotenuse* slider. What happens to $m\angle CAB$? What happens to $\sin A$?

6. Construct the other two ratios based on *SOH CAH TOA*. Label the ratios *cos A* and *tan A*.

What term is used for triangles of different size but the same shape?

Q4 Why do the trigonometric functions stay the same if the triangle is made larger or smaller without changing the angles?

ANGLES UP TO 90°

You have just determined how the triangle's size affects (or does not affect) the values of the ratios. Now you'll make a table and graph to investigate how the angle affects the ratios.

7. Place $m\angle CAB$, $\sin A$, $\cos A$, and $\tan A$ in a table by selecting all four values in order and choosing **Graph | Tabulate.**

8. Change the angle and watch the numbers change. Double-click the table to make the first row permanent. Continue dragging *B* and double-clicking the table until you have ten rows of values for different angles between 0° and 90°.

9. Press *Show Axes.* Then select the table and choose **Graph | Plot Table Data.** Choose $m\angle CAB$ for *x* and $\sin A$ for *y*.

Q5 On your paper, sketch the shape of the plotted points.

To plot a point, select two measurements in order and choose **Measure | Plot As (x, y).**

To trace a point, select it and choose **Display | Trace Plotted Point.**

10. To get a more complete graph of the relationship, plot the point ($m\angle CAB$, $\sin A$). Turn on tracing for the plotted point.

11. Fill in the gaps of your graph by dragging *B* to change the angle from 0° to 90°.

12. Following the same steps, plot both ($m\angle CAB$, $\cos A$) and ($m\angle CAB$, $\tan A$). Sketch and label all three graphs on your paper.

You can use the buttons to set the angle exactly.

Q6 Describe the behavior of each graph. Is it increasing, decreasing, or constant? For what angles is it changing quickly? For what angles is it changing slowly?

Q7 Does $\cos 90° = \cos 30° + \cos 60°$? Use your cosine graph to explain your answer.

As a hint, set angle *A* to 60°. What is angle *B*? Look at the sides involved for cos *A* and sin *B*.

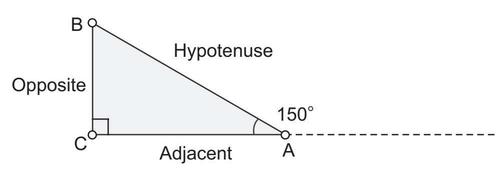
Q8 Why does $\cos 60° = \sin 30°$? Use your answer to find a second pair of angles for which the cosine of one is equal to the sine of the other.

Q9 What happens to the tangent when the angle is 90°?

EXPLORE MORE

Since a right triangle cannot have an obtuse angle, can $\sin 150°$ exist? What would the triangle look like if you could drag *B* so that $m\angle CAB > 90°$? What would happen to the ratios?

Q10 Why can a right triangle not have an obtuse angle?



Q11 On page 2 point *B* is free to move in a complete circle. What happens when you drag the angle past 90°? How could you use the result to define $\sin 150°$?

13. Using Sketchpad's Calculator, calculate $\sin 150°$ and $\sin 210°$.

Q12 How do these values compare to $\sin 30°$?

Q13 Find another angle greater than 180° whose sine is the same as $\sin 210°$.

Objective: Students calculate various ratios for right triangles. By plotting the resulting values, they see the graphs of the trigonometric functions emerge for angles between 0° and 90°. Finally, students are challenged to think about the values of the ratios for angles beyond 90°.

Student Audience: Algebra 2

Prerequisites: None

Sketchpad Level: Intermediate. Students measure lengths and angles, change object labels, tabulate values, and plot points.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (**Right Triangle Functions.gsp**) or Whole-Class Presentation (use **Right Triangle Functions Present.gsp**)

RATIOS

Q1 $\sin 30° = \frac{1}{2}$. If students drag point B to make the angle 30°, the value could be slightly different. By using the button, they will get the exact value.

Q2 Answers will vary. The important thing is that students make predictions before trying it.

Q3 As students change the slider, the size of the triangle changes, but the angle and ratio remain constant.

Q4 The angles for the triangle remain constant as the size of the triangle changes. The resized triangle is similar to the original, so there is a scale factor by which each side has been multiplied to generate the new triangle. Therefore, each ratio must remain constant, because its numerator and denominator have been multiplied by the same factor.

ANGLES UP TO 90°

Q5 Students should end up with a sketch of the sine function between 0° and 90°.

12. It's important that students sketch the graphs themselves on paper, rather than printing out the sketch. The physical act of sketching the shapes helps them to remember the characteristics of the graphs.

Q6 The value of the sine increases from 0° to 90°, quickly at first and then more and more slowly. The value of the cosine decreases from 0° to 90°, slowly at first and then more quickly. The value of the tangent increases from 0° to 90°, very quickly as it approaches 90°.

Q7 No, $\cos 90° \neq \cos 30° + \cos 60°$. A reason can be found by looking at the plot for cosine. Using the plot, you can see that the value of $\cos 30° + \cos 60°$ is definitely greater than the value of $\cos 90°$. In general, there are very few functions where $f(x+y) = f(x) + f(y)$.

Q8 The values of $\cos 60°$ and $\sin 30°$ can be represented in one triangle. If $\angle A$ is 60°, $\angle B$ must be 30° since this is a right triangle. This illustrates that cofunctions of complementary angles are equal, because they refer to the same parts of the triangle.

Q9 The tangent value is undefined. The length of the adjacent side is 0, and division by zero is undefined.

EXPLORE MORE

Q10 If the triangle had an obtuse angle and a right angle, the sum of the angles would be more than 180°.

Q11 When you move B past the vertical to try to make an angle of 150°, you end up with a right triangle with an angle of 30°. (This angle is called the *reference angle* for 150°.) You could use this result to define $\sin 150° = \sin 30°$. (Although this result is correct, $\cos 150° \neq \cos 30°$, because the signs are different. Proper definitions of these functions for angles over 90° is best done by using the unit circle.)

Q12 The Calculator results indicate that $\sin 150° = \sin 30°$ and that $\sin 210° = -\sin 30°$. Students may speculate that the difference is that the opposite side goes up for 150° but down for 210°.

Q13 Another angle with the same sine is 330°.

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Right Triangle Functions Present.gsp** to present this activity to the whole class.

Use this presentation to review the right triangle definitions of the trigonometric functions and extend them to angles greater than 90°.

1. Open **Right Triangle Functions Present.gsp.** The measure of $\angle CAB$ is on the screen. Drag point B to show its range.

Q1 For $\angle CAB$, identify the opposite side, adjacent side, and hypotenuse.

2. Press *Show Labels* and *Show Lengths.* Review the sine, cosine, and tangent definitions.

Q2 Set $\angle CAB$ to an arbitrary angle. Which of the three ratios is the greatest? Which is the least? (The answer depends on the angle. Students should be able to make a good guess by looking at the sides.) Press *Show Ratios* to reveal the answers. Repeat this part.

3. Pick another arbitrary angle. Have a student write the three ratios on the board.

Q3 What is the complement of this angle? What are the sine, cosine, and tangent of the complement?

4. Press the *Complement* button and discuss the relationships between the ratios for the original angle and the ratios for the complement.
5. Go to page 2. Again, you can change the angle by dragging a point.
6. Press *sin x* to see the point (*x, sin x*). Change the angle to create a trace of the sine graph. Do the same for the cosine and the tangent.

Q4 What is the geometric relationship between the sine and cosine graphs, and how is this explained by the relationship $\sin x = \cos(90° - x)$?

Q5 Why is there no upper limit to the tangent function?

Page 3 introduces trigonometric functions for angles greater than 90°. Discuss the fact that although an angle can be greater than 90°, an angle of that size will not fit into a right triangle. In that case, there is a related acute angle that serves as a *reference angle.* On page 3, $\angle CAB$ is the reference angle for $\angle DAB$.

7. Press *Show Angle CAB* and *Show Angle DAB.* Drag point B around to change the angles.

Q6 How are the sine, cosine, and tangent of these angles related? Students will see that the functions of $\angle DAB$ are sometimes negative, but always have the same magnitude as the corresponding functions of $\angle CAB$. Keep dragging point B and guide the class in determining the domain in which each function is negative.

8. On page 4 trace the function graphs again, this time for all angles between 0° and 360°.

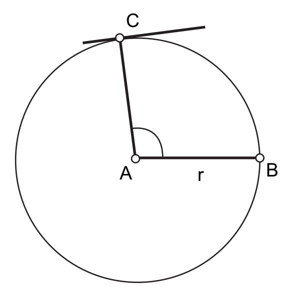
Radian Measure

So you think you know angle measurement? An understanding of degrees is a valuable skill, but there are other ways to measure. Other angle units include points, grads, mils, and dekans. To make matters worse, these units may go by different names in different places. Still worse, they may have the same name, but different definitions.

Above all of these angle units, the radian holds a special place. You can use it to measure angles, of course, but radian measure also describes relationships between certain geometric objects.

WHAT IS A RADIAN?

1. Open **Radians.gsp.** Press the *Go* button and watch the circle radius rotate into a tangent position and then roll around the circle.



You will use this line segment to measure a central angle of the circle. This will be the basis for defining radian measure for angles.

2. Press *Reset* and then *Home* to stop the animation and return the radius to its tangent position. Measure the radius of the circle and the length of the blue segment.
3. Press the buttons *Show Central Angle, Show Arc,* and *1 Radian.* Measure the length of arc *a*.

Q1 The central angle, $\angle BAC$, is now exactly one radian. What do you notice about your three measurements? Explain why they come out this way.

Notice that although *theta* is equivalent to $m\angle BAC$, it can keep increasing past 360°.

Q2 The measure of the angle, *theta*, is displayed in degrees. Approximately how many degrees are there in one radian?

4. Press *Semicircle.* The line segment will continue to roll until it has stepped off half of the circle.

Q3 Using the tick marks to approximate an answer, how many radians are in a semicircle? How many radians will there be in a complete circle?

5. Press the *1 Circle* button to check your last answer.
6. Choose **Edit | Preferences.** Change the Angle Units to **radians.**

Q4 The angle measurement now shows you exactly how many radians are in a circle. But you already knew that, didn't you? Write the formula for the circumference in terms of the radius. Use that along with the definition of a radian to prove that there are exactly 2π radians in a circle.

WHY RADIANS?

So far, you have not seen any good reason for using radians rather than degrees. Actually, we use radians in order to make things easier, not harder.

Do these measurements one at a time. Select one object and choose the appropriate command from the Measure menu.

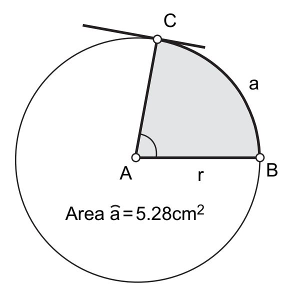
To change the radius, drag point *B*.

7. Press *Reset* and *Go*. Press *Go* again to stop the animation before the angle makes a complete circle ($0 < theta < 2\pi$).
8. You have a measurement for angle *theta* and a measurement for radius *r*. Use the calculator to find the product *theta* · *r*.

Q5 What is the arc length in terms of *theta* and *r*? Check your answer with different radii and different angles in the range $0 < theta < 2\pi$. Does your formula always work? Does it work when you use degrees?

The area of a circle sector varies directly with the central angle. You probably are familiar with this formula:

$$\text{sector area} = \frac{\theta}{360°}\pi r^2$$



Q6 Rewrite the above formula using radians instead of degrees. Simplify your answer.

9. Select the arc and choose **Construct | Arc Interior | Arc Sector.** Select the sector and choose **Measure | Area.**

Q7 Using your formula from Q6, calculate the area of the sector. Does it match your measurement in all cases?

DISCUSS

Q8 When using radians, Sketchpad automatically expresses angle measurements in multiples of π. This is a common practice. Why?

Q9 It is also common practice (not used by Sketchpad) to write radian angle measurements without writing any units at all. Why is that?

Q10 In spite of the radian advantages you have seen here, degrees are more common in practical applications. What advantages do degrees have?

Objective: Students explore the relationship between the length of a circular arc, its radius, and its central angle measured in radians.

Student Audience: Geometry/Algebra 2/Precalculus

Prerequisites: Students need no previous experience with radians. However, if this is an introduction, it would be best to show them a sketch of a one-radian sector and explain that the arc length is equal to the radius.

Sketchpad Level: Intermediate. Students must do a few measurements and calculations.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Radians.gsp**) or Whole-Class Presentation (use **Radians Present.gsp**)

WHAT IS A RADIAN?

Q1 All three measurements (the radius of the circle, the length of the blue segment, and the length of arc *a*) are the same. This is because the blue segment started out as a radius of the circle, and when it rolled along the circle, it measured out an arc that is the same length as it is.

Q2 1 radian ≈ 57.30°

Q3 A semicircle has a central angle of exactly π radians. If students are not aware of that fact, they should be able to give an estimate between 3.0 and 3.3. There are 2π radians in a complete circle.

Q4 Start with the circumference formula:

$$\text{circumference} = 2\pi r$$

This means that there are 2π radius lengths in the circumference, so the angular measure is 2π radians.

WHY RADIANS?

Q5 The measurements should verify this fundamental relationship:

$$\text{arc length} = \theta r$$

The formula will work for any angle in the given range, and for any radius. It will not work when the angle units are degrees.

Students may notice some conflicts with the units. The arc length will be in centimeters, but the θr calculation will be in radians • centimeters. Leave that discrepancy for the discussion.

Q6 When degrees are converted to radians, you get this simpler formula:

$$\text{area} = \frac{\theta}{360°}\pi r^2 = \frac{\theta}{2\pi}\pi r^2 = \frac{\theta r^2}{2}$$

Q7 The formula works for any angle between 0 and 2π, and for any radius, but it does not work when the angle units are degrees. Again, the number values agree, but there is an apparent discrepancy with the units.

DISCUSS

The discussion questions will be more helpful if the entire class works together. Here are some suggested points.

Q8 Although radian angle measurement is useful, an angle of one radian has no great significance. The really useful angles (180°, 90°, 60°) have measures that are irrational numbers when expressed in radians, so they cannot be expressed exactly with a decimal expansion. However, we can express them as simple fraction multiples of π (π, $\pi/2$, $\pi/3$).

Q9 A radian measurement is an angle measurement, but you can just as well think of it as the ratio of the lengths of an arc and its radius. Since it is a ratio of two linear measurements, it has no units. An advantage of this concept is that it clears up the unit discrepancy that appeared in Q5 and Q7.

Q10 The number of radians in a circle is 2π, an irrational number. It is impossible to divide a circle into an integral number of radians. It is also impossible to do this with a tenth, hundredth, thousandth, or any other fraction of a radian. Instruments that measure angles (protractors, compasses, theodolites, sextants) need to have divisions that are all the same, so the angle unit must divide the circle evenly.

One obvious solution might be simply to graduate the instruments in some fraction of π radians. In fact, that is exactly what we do ($1° = \pi/180$ radians).

1. Open **Radians Present.gsp.** Press the *Go* button.

As the radius segment rolls around the circle, explain that students can think of a radian as the angle that corresponds to one length of the radius being laid out along the circumference of the circle.

Q1 How long is the blue segment? (It's equal to the radius of the circle.)

Q2 What do the red ticks mark off? (Each tick marks a distance of one radius and an angle of one radian.)

2. Press *Reset,* then *1 Radian.* The animation stops after marking off one radian.

Q3 About how many degrees are there in one radian? To coax a good guess, point out the triangle formed by points *A, B,* and *C.* You can think of arc *BC* as a side of the triangle. That would make it an equilateral triangle, but one of the sides is not straight. Would that make $\angle BAC$ greater than 60° or less than 60°?

3. Press *Show Central Angle.* It will show that *theta* ≈ 57.30°.
4. Press *Semicircle.*

Q4 Count the tick marks. How many radians are there in a semicircle? (a little more than 3)

To change the angle units, choose **Edit | Preferences.**

5. Change the Angle Units to radians. Angle *theta* appears as 1π radians.
6. So a semicircle has π radians. That means that a circle must have 2π radians. Press *1 Circle* to confirm that.

Q5 But you already knew that, didn't you? What is the circumference in terms of *r*? ($2\pi r$) So how many times will the radius go into the circumference? (2π) So how many radians are there in a circle? (2π)

7. Press *Reset,* then press *Go.* Press *Go* again to stop the animation with *theta* somewhere between 0 and 2π. Press *Show Arc.*

When entering numbers that are on the screen, click on the measurement itself.

8. Here is something you can do with radians, but not with degrees. Choose **Measure | Calculate.** Enter *theta* · *r*. Compare the calculation with the measured length of the arc. Try it with several different values of *theta* and *r*.

Q6 The formula for area of a sector is $\frac{\theta}{360°}\pi r^2$. Convert the 360° to radians and simplify. What is the new formula? $\left(\frac{\theta r^2}{2}\right)$

9. Press *Show Sector*. Use the Sketchpad Calculator and enter *theta* · $r^2/2$. Compare the answer to the measured arc length.

Unit Circle Functions

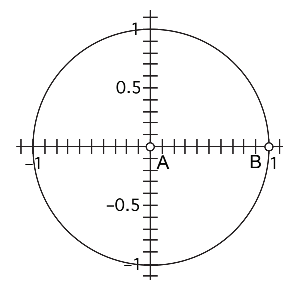
There are several different ways of defining trigonometric functions like sine and cosine. One set of definitions is based on right triangles, but right triangle definitions are limited to angles between 0 and $\pi/2$. (Recall that $\pi/2 = 90°$.) In this activity you'll use a *unit circle* (a circle with a radius of exactly one unit) to define trigonometric functions for any possible angle, even beyond 2π.

CONSTRUCT A UNIT CIRCLE

Start by creating a coordinate system, constructing a unit circle, and making some measurements.

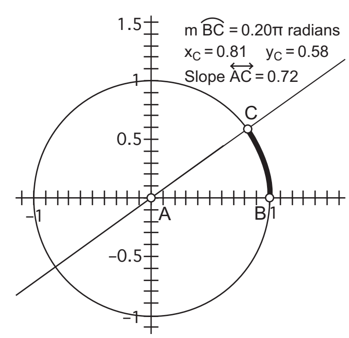
Use the Units panel of Preferences to set units and precision, and use the Color panel to turn on trace fading.

1. In a new sketch, set the Angle Units to **radians,** set the Precision for slopes and ratios to **thousandths,** and turn on trace fading. Use the Preferences dialog box to make all three of these settings, by choosing **Edit | Preferences.**



2. Choose **Graph | Show Grid** and resize the axes (by dragging the number on one of the tick marks) so that the maximum x-value is between 6 and 7.
3. Label the origin A and the unit point B by selecting them in order and choosing **Display | Label Points.**
4. Construct a unit circle. With points A and B still selected, choose **Construct | Circle By Center+Point.**
5. Construct a point on the circle and label it C. (Be sure you don't construct it where the circle intersects one of the axes.)
6. Measure the x- and y-coordinates of this new point separately. Choose **Measure | Abscissa (x)** and **Measure | Ordinate (y).**

Use the **Line** tool, or else select points A and C and choose **Construct | Line.**



7. Construct a line through the origin and the point that you just labeled.
8. Measure the slope of this line by choosing **Measure | Slope.**

To construct the arc, select the circle and points B and C in order. Then choose **Construct | Arc On Circle.** Choose **Display | Line Width | Thick** to make it thick.

9. On the circle, construct an arc that begins at the x-axis (at unit point B) and goes counter-clockwise to point C. Make the arc thick.
10. With the arc still selected, measure its arc angle.
    Drag point C around the circle and observe how all four measurements behave.

Q1 What are the largest and the smallest values you observe for each measurement? Where do you find these largest and smallest values?

PLOT YOUR MEASUREMENTS

To explore how the measured quantities depend on the position of point *C,* you'll plot each measurement using the arc angle as the independent variable.

To plot the point, select in order the independent variable (the arc angle) and the dependent variable (the *y*-coordinate), and choose **Graph | Plot As (x, y).**

You may want to turn off tracing for your first plotted point before looking at the second one.

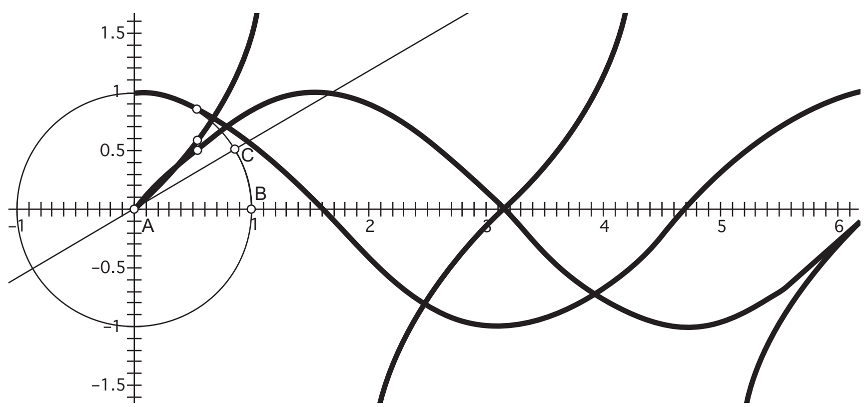
11. Plot the *y*-coordinate of point *C* as a function of the arc angle. With the plotted point selected, choose **Display | Trace Plotted Point.**

Q2 Examine the trace that appears as you drag point *C* around the circle. Describe its shape as you drag point *C* through the four quadrants. Do you recognize this graph? Which trigonometric function is this?

12. Plot the *x*-coordinate of point *C* as a function of the arc angle. Turn on tracing for this plotted point, and then drag *C* to observe how it behaves.

Q3 Describe the shape of this trace as you drag point *C* through the four quadrants. Which trigonometric function is this?

13. Plot the slope of the line as a function of the arc angle. Turn on tracing, drag *C*, and observe the result.



Q4 Describe the shape of this trace as you drag *C* through the four quadrants. Which trigonometric function is this?

Q5 Calculate the value of y_C/x_C. Compare this value to the value of the slope while you drag point *C*. What do you notice? Explain your observations.

EXPLORE MORE

Q6 Through point *B,* create a tangent to the unit circle by constructing a line perpendicular to the *x*-axis. Construct the intersection of this tangent line with the line through points *A* and *C,* and measure the coordinate distance from the point of tangency to this intersection. How does this measurement compare with other measurements you have made? How does this measurement help explain the name of one of the trigonometric functions?

The unit circle allows you to use angles greater than $\pi/2$, but by using an arc, you are still limited to positive angles less than 2π. To use angles outside this domain, change point *C* so it's a rotated image of point *B*.

14. Create an angle parameter by choosing **Graph | New Parameter.** Name the angle *theta* and set its units to radians.
15. Mark point *A* as the center of rotation by selecting it and choosing **Transform | Mark Center.**
16. Rotate point *B* by the value of *theta* by selecting it and choosing **Transform | Rotate.** When the Rotate dialog box appears, click *theta* in the sketch to use the parameter as the angle of rotation.
17. Split point *C* from the circle and merge it to the rotated image. (Select *C* and choose **Edit | Split Point From Circle.** Then select both *C* and the rotated image, and choose **Edit | Merge Points.**)

The value of the parameter changes by $\pi/36$ (the equivalent of 5°) every time you press the **+** or **–** key. By using only these keys to change the parameter, you can keep its value at round numbers.

Q7 Set *theta* to zero, and then press the + key repeatedly to change the angle. Record at least four angles for which the slope is zero. Then use the − key to find two more angles (less than zero) for which the slope is zero.

Q8 Find three different angles for which the y-value of the point on the circle is 1. At least one of your angles should be negative.

Q9 Find one angle for which the x-value of the point on the circle is -1. Then write down five more such angles, without actually trying them.

Q10 Find one angle for which the y-value of the point on the circle is 1/2. Then write down five more such angles.

Objective: Students use a unit circle to define the trigonometric functions. They construct a point on the unit circle, measure the coordinates of the point, and graph the coordinates as a function of the arc angle from the positive x-axis to the point, producing plots of the sine and cosine functions. They also plot the slope of the line from the origin through the point, producing a plot of the tangent function.

Student Audience: Algebra 2/Precalculus

Prerequisites: None. You can use this activity as the first definition of the trigonometric functions, or you can use it after you've already defined the functions in a triangle.

Sketchpad Level: Intermediate. Students perform the construction and measurements themselves.

Activity Time: 35–45 minutes

Setting: Paired/Individual Activity (no sketch needed) or Whole-Class Presentation (use **Unit Circle Functions Present.gsp**)

Related Activity: Unit Circle and Right Triangle Functions

Most teachers define the trigonometric functions in triangles before introducing the unit circle, but others may prefer to present the unit circle definitions first. This activity supports either way of introducing these definitions.

If you are using the activity as the original definition of the functions, you should introduce it by telling students that the purpose of the activity is to define three new functions. Tell them that the function they will describe in Q2 is the sine function, the function they will describe in Q3 is the cosine function, and the function they will describe in Q4 is the tangent function.

If students have already learned the definitions of these three functions in right triangles and have seen graphs of the functions, they should be able to relate the graphs in Q2, Q3, and Q4 to the functions they already know.

In either case it's important to relate the definitions developed in this activity to the definitions in the right triangle. The activity Unit Circle and Right Triangle Functions is an excellent way to make this connection.

CONSTRUCT A UNIT CIRCLE

10. If students measure $\angle BAC$, the measurement will be shown as a value from $-\pi$ to π. By constructing an arc instead and measuring the arc angle, students end up with a measurement from 0 to 2π, and their graph shows a complete period starting at 0.

Q1 The value of x_C ranges from -1 at the left edge of the circle to $+1$ at the right edge. The value of y_C ranges from -1 at the bottom of the circle to $+1$ at the top. Answers for the slope of line AC will vary depending on how close students can drag point C to the precise bottom of the circle (where they find the smallest value, probably between -1000 and -100) and the precise top (where they find the largest value, probably between 100 and 1000). The arc angle varies from 0 at the right edge of the circle to 2π, also at the right edge of the circle.

PLOT YOUR MEASUREMENTS

Q2 The trace starts at the origin, curves up to its maximum of 1 as C leaves Quadrant I, curves back down to 0 as C leaves Quadrant II, continues to -1 as C leaves Quadrant III, and finally curves back to 0 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the sine function. If you are using this activity to introduce the functions, you will need to tell them that the y-coordinate on the unit circle defines the sine function.

Q3 The trace starts at (0, 1), curves down to 0 as C leaves Quadrant I, continues to -1 as C leaves Quadrant II, curves back to 0 as C leaves Quadrant III, and continues on to 1 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the cosine function. If you are using this activity to introduce the functions, you will need to tell them that the x-coordinate on the unit circle defines the cosine function.

Q4 The trace starts at the origin, curves up and shoots off the top of the screen as C leaves Quadrant I, reappears from the bottom of the screen and curves up to 0 as C leaves Quadrant II, continues once more off the top of the screen as C leaves Quadrant III, and reappears again at the bottom of the screen and curves back to 0 as C reaches the end of Quadrant IV.

If students have already seen graphs of the trigonometric functions, they should recognize this as the graph of the tangent function. If you are using this activity to introduce the functions, you will need to tell them that the slope from the origin to the point on the unit circle defines the tangent function.

Q5 The calculation y_C/x_C gives the same result as the slope measurement, because it corresponds to the definition of slope: rise/run or $\Delta y/\Delta x$. Use students' answers to this question to make the point that the tangent function can be defined in three equivalent ways: as the slope, as y/x, or as $\sin\theta/\cos\theta$.

EXPLORE MORE

Q6 The coordinate distance from the point of tangency to the intersection is the same as the slope measurement. Applying the slope formula to the point of intersection (x, y) and the origin $(0, 0)$ results in y/x. Because the x-coordinate along the tangent line is 1, the y-coordinate of the intersection must be equal to the slope; that is, it's equal to the value of the tangent function. Thus, the length of the tangent from the point of tangency $(1, 0)$ to this intersection is the value of the tangent function. It's this fact that originally gave the tangent function its name.

The unit circle extends the domain of the trigonometric functions to 2π; but to go beyond 2π, students need to consider the possibility of wrapping around the unit circle more than once. The remaining three questions require them to do so, and also to consider negative angles as angles measured in the opposite (clockwise) direction.

Q7 Angles for which the slope (tangent function) is 0 include $0, \pi, 2\pi$, and 3π. The angle $-\pi$ is the first angle less than 0 for which the slope is 0. The general form is $n \cdot \pi$ for any integer n.

Q8 The y-value (sine function) of the point on the circle is 1 for positive angles $\pi/2$, $5\pi/2$, $9\pi/2$, and so forth. It's also 1 for negative angles $-3\pi/2$, $-7\pi/2$, and so forth. The general form is $\pi/2 + 2\pi n$ for any integer n.

Q9 The x-value (cosine function) of the point on the circle is -1 for positive angles π, 3π, 5π, and so forth. It's also -1 for negative angles $-\pi$, -3π, -5π, and so forth. The general form is $\pi + 2\pi n$ for any integer n.

Q10 The y-value (sine function) of the point on the circle is 1/2 for positive angles $\pi/6$, $5\pi/6$, $13\pi/6$, $17\pi/6$, and so forth. The general forms are $\pi/6 + 2\pi n$ and $5\pi/6 + 2\pi n$ for any integer n.

WHOLE-CLASS PRESENTATION

Use the Presenter Notes and **Unit Circle Functions Present.gsp** to present this activity to your class.

Use this presentation to define the trigonometric functions based on the unit circle. You can use this as the fundamental introduction to these functions, or you can use it after presenting the definitions in right triangles.

PRESENT

1. Open **Unit Circle Functions Present.gsp.**

Q1 Ask students for the radius of the circle. (1)

2. Press the *Animate C* button to move *C* around the circle. Make sure that students notice arc *BC*.

Q2 Show the arc angle measurement, and ask students to determine the smallest and largest values for this measurement. (0 and 2π)

Do not actually measure the arc length. The measured length would not be based on the coordinate system unit.

Q3 Stop the animation and ask students what the arc length is. (It's the same as the arc angle, because the radius is 1.)

Q4 Restart the animation and show the *y*-coordinate. Ask students to observe and determine the smallest and largest values, and where they occur. (The value of y_C ranges from -1 at the bottom of the circle to $+1$ at the top.)

Q5 Tell students you are about to plot the *y*-coordinate as a dependent variable. Ask them what measurement to use as the independent variable. (arc angle *BC*)

3. Stop the animation and press *Show Plotted y-value* to see the plotted point. Drag *C* so students can verify that the plotted point matches their observations from Q4. Then restart the animation to get a smooth plot of the traced point.

Q6 What trigonometric function does this plot represent? (It is the sine function.)

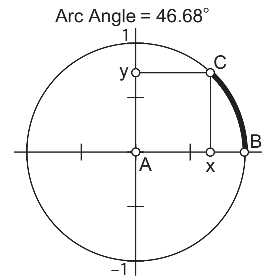
4. Repeat the same actions and questions for the *x*-value (resulting in a plot of the cosine function) and the slope measurement (resulting in a plot of the tangent function).

5. Ask students for the domain of the plots they have seen. (0 to 2π) Then go to page 2 and use the buttons to show how the domain can be extended to negative angles and to angles greater than 2π.

Unit Circle and Right Triangle Functions

There are several different ways to define trigonometric functions like sine and cosine. One set of definitions is based on right triangles, and another set is based on a *unit circle* (a circle with a radius of exactly one unit). In this activity you'll explore the relationship between these two ways of defining trigonometric functions.

THE UNIT CIRCLE



1. Open **Unit Circle Right Triangle.gsp.** Measure the arc angle of arc *BC* on the unit circle. Label the measurement *Arc Angle.*

Q1 Drag point *C* around the circle and observe the angle measurement. What are the smallest and largest values that you observe? Leave *C* in Quadrant I when you finish.

2. Measure the *y*-coordinate of point *C* and label it *sin in circle*. Measure the *x*-coordinate and label it *cos in circle.*

3. Construct a line through *A* and *C,* and measure the slope of the line. Label this measurement *tan in circle.*

Q2 Drag *C* again. What are the smallest and largest values that you observe for the sine, cosine, and tangent of the arc angle? At what angles do these values occur?

THE REFERENCE TRIANGLE

Ratios of triangle sides provide another way to define trigonometric functions. You can use the mnemonic *SOH CAH TOA* to recall the ratios:

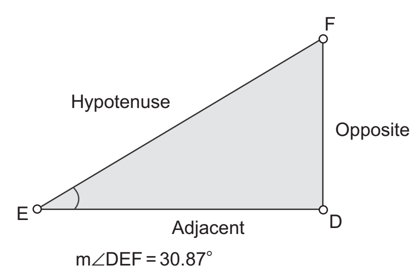
SOH: The ratio for *Sine* is *Opposite* over *Hypotenuse.*

CAH: The ratio for *Cosine* is *Adjacent* over *Hypotenuse.*

TOA: The ratio for *Tangent* is *Opposite* over *Adjacent.*

To measure ∠*E*, select points *D*, *E*, and *F*. Then choose **Measure | Angle.**

4. Measure ∠*E* for the right triangle.



5. Measure the *Adjacent* side by selecting points *D* and *E* and choosing **Measure | Coordinate Distance.** Do the same for the other two sides. Label your measurements *Opposite, Adjacent,* and *Hypotenuse.*

6. Use Sketchpad's Calculator to calculate *Opposite*/*Hypotenuse.*

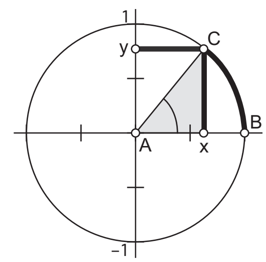
Q3 According to the *SOH CAH TOA* mnemonic, to which trigonometric function does this calculation correspond?

7. Label your calculation *sin in triangle*. Calculate each of the other two ratios and label them appropriately.

Q4 Drag point *F*. What are the smallest and largest values that you observe for the sine, cosine, and tangent in the right triangle? At what angle do the maximum and the minimum occur for each?

COMPARE THE DEFINITIONS

To compare these definitions, you'll combine the two models.



8. Select points *A* and *E,* and choose **Edit | Merge Points.** Also merge points *C* and *F*. The right triangle is now attached to the inside of the unit circle.

Q5 Drag point *C* and observe the two angle measurements (the arc angle and the angle in the triangle). When do these measurements agree? When do they disagree?

Q6 Drag point *C* and observe the two sine measurements. Explain why the values are equal in certain quadrants but not in others.

Q7 When do the two cosine measurements agree, and when do they disagree? Why?

Q8 When do the two tangent measurements agree? Explain.

Q9 Why is the sine of 150° the same value as the sine of 30°? Why is the sine of 210° the opposite of the sine of 30°? (*Hint:* Think about how each relates to either a coordinate or a ratio, and compare these.)

Q10 Describe possible advantages and disadvantages for each method of defining the trigonometric functions.

EXPLORE MORE

Q11 Based on the different definitions, which might be better to determine the flight path of an airplane? The position of a person on a Ferris wheel? The height of a building? Explain.

Q12 Could you always use a single definition? Explain.

Q13 Drag point *C*. What happens to the tangent at 90°? Explain. What does this mean in terms of the graph of the tangent function at 90°?

Objective: Students compare the unit circle definitions of trigonometric functions with the right triangle definitions. They combine the two models and examine similarities and differences that emerge.

Student Audience: Algebra 2/Precalculus

Prerequisites: It's best (though not necessary) if students have seen both sets of definitions of the trig functions. (The Related Activities listed below cover this ground.)

Sketchpad Level: Intermediate. Students measure values, and merge points to combine the two models.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Unit Circle Right Triangle.gsp**) or Whole-Class Presentation (use **Unit Circle Right Triangle Present.gsp**)

Related Activities: The definitions are introduced in Right Triangle Functions and in Unit Circle Functions.

THE UNIT CIRCLE

Q1 The value of *Arc Angle* ranges from 0° to 360°.

Q2 The sine ranges from −1 to 1, at angles of 270° and 90°, respectively. The cosine ranges from −1 to 1, at angles of 180° and 0°, respectively. The tangent has no limit in either direction, but the measurement is limited by the resolution of the objects on the screen.

THE REFERENCE TRIANGLE

Q3 The calculation corresponds to the sine function.

Q4 The smallest value for sine is 0 and occurs at 0°. The largest value is 1 and occurs at 90°. The smallest value for cosine is 0 and occurs at 90°. The largest value is 1 and occurs at 0°. The smallest value for the tangent occurs at 0°. The tangent has no upper limit, and it gets very large as the angle approaches 90°. At both 0° and 90°, the triangle is degenerate, with various points and sides coinciding.

COMPARE THE DEFINITIONS

Q5 The measurements agree only in the first quadrant. In the other quadrants the arc angle is more than 90°, but the angle in the triangle remains between 0° and 90°.

Q6 The definitions agree in Quadrants I and II because the y-value is positive there. The definitions disagree in the other two quadrants because the measured length of a line segment is always positive.

Q7 The cosine values agree in Quadrants I and IV, but disagree in Quadrants II and III. In these two quadrants the x-value is negative, but the distance measured in the triangle remains positive.

Q8 The tangent values agree in Quadrants I and III. In Quadrant I the coordinates (for the unit circle definition) and the distance measurements (for the right triangle definition) are all positive, so the two functions agree. In Quadrant III both coordinates are negative, so their ratio is positive, matching the right triangle definition. In the other two quadrants one coordinate or the other is negative, resulting in values that the right triangle cannot produce.

Q9 Explanations will vary. This is a good place to introduce the idea of the *reference triangle* within the unit circle and to observe that the opposite side for both 30° and 150° corresponds to the same y-value. For 210°, the opposite side corresponds to a negative y-value, so the value of $\sin 210°$ is the opposite of that of $\sin 30°$.

Q10 Answers will vary. A big advantage of the unit circle method is the ability to work with angles that are beyond 90°. An advantage of the right triangle method is that it's easier to apply when the angle is not in standard position. (Though students don't know this yet, the unit circle method will allow them to explore topics, such as uniform circular motion, which would not be possible with only a right triangle definition.)

EXPLORE MORE

Q11 Answers will vary. Analyzing the flight path of a plane or the position of a person on a Ferris wheel both benefit from using angles beyond 90°. For the height of a building, a right triangle definition is sufficient.

Q12 You could condense the two methods into one by considering the right triangle method to be a special case of the unit circle in Quadrant I.

Q13 At 90° the line AC is vertical, so its slope (and the tangent of 90°) is undefined. The result is that the tangent graph has an asymptote at 90°.

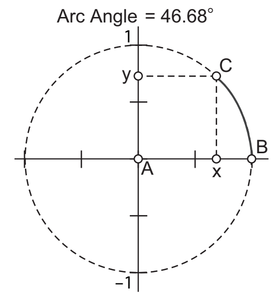
Unit Circle and Right Triangle Functions Presenter Notes

Use this presentation to relate the two ways of defining the trig functions.

THE UNIT CIRCLE

Angles in this sketch are in degrees rather than radians.

Leave point *C* in motion while students answer these three questions.



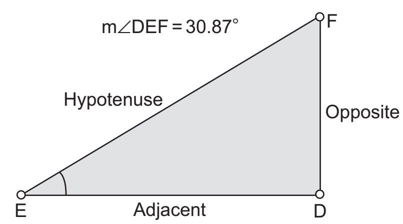
1. Open **Unit Circle Right Triangle Present.gsp.** Show the unit circle and animate point *C*. Show the arc angle.

Q1 Ask students what are the largest and smallest values they observe for the arc angle. (0° to 360°)

Q2 Ask what measurements are needed in the unit circle to define the sin, cos, and tan functions. (in order: *y*, *x*, and equivalently either *y*/*x* or the slope of *AC*)

Q3 Show these measurements and ask students to observe the largest and smallest values for each of the measurements. Review again which is sin, cos, and tan.

THE REFERENCE TRIANGLE



2. Show the right triangle and measure ∠*DEF*.

Q4 Drag point *F* and ask students to observe the largest and smallest values for the angle.

Q5 Show the length measurements and ask students what ratios must be calculated to find the sine, cosine, and tangent.

Q6 Show the ratios and have students confirm which is sine, cosine, and tangent. Drag point *F* and have students observe the largest and smallest values for each ratio.

COMPARE THE DEFINITIONS

3. To compare the definitions, combine the models. Press *Merge Triangle to Circle.*

Q7 Drag point *C* (keeping it in Quadrant I), and ask students to compare the four measurements from each triangle.

Q8 Ask students to make conjectures about what will happen if *C* leaves Quadrant I.

Q9 Drag point *C* slowly through the other three quadrants, and ask students to describe what they observe about each of the four measurements. Encourage them to explain their observations.

Q10 Why does the sine of 150° in the circle have the same value as the sine of 30° in the triangle? Why is sin 210° the opposite of sin 30°?

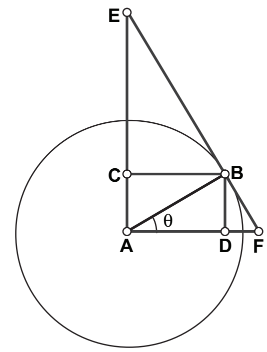
Q11 Ask for advantages and disadvantages of each way of defining the functions.

Trigonometric Identities

When you first learned trigonometry, you probably used only acute angles. That's because you can put an acute angle into a right triangle and all of the trigonometric functions will be ratios of sides. Even as you get into more advanced concepts, it can be helpful to keep going back to the right triangle geometry.

ACUTE ANGLES

1. Open the first page of **Trigonometric Identities.gsp.**



This sketch shows acute angle θ at the center of a unit circle. Controls at the bottom allow you to change the angle or the radius. (When you change the radius, all you change is the scale. The radius of the circle is always one.)

2. To get a feel for the construction, drag the angle control at the bottom of the screen.

Segment AB is the terminal side of angle θ. Segment EF is tangent to the unit circle at point B. The other line segments are either horizontal or vertical.

Q1 Notice that angle θ appears in several places, and there are also a number of right angles. Triangle ABD is a right triangle with angle θ. How many triangles are similar to $\triangle ABD$? Identify as many of them as you can.

The measure of angle θ is on the screen, so you can confirm these answers by calculating the trigonometric functions of θ.

Q2 Along the edge of the screen are length measurements for eight line segments. By forming a ratio with AB, you can show that each length is equal to a trigonometric function of θ. Follow this example:

$$\sin\theta = \frac{BD}{AB} = \frac{BD}{1}, \text{ so } \sin\theta = BD$$

For each remaining length measurement, find the corresponding trigonometric function. Some functions appear more than once.

Q3 Lengths of corresponding sides of similar triangles are proportional, so you can write many proportions in this figure. Here is one:

$$\frac{BD}{AB} = \frac{BF}{AF}$$

Using your answers from Q1 and Q2, substitute trigonometric functions into this proportion. What identity do you get?

Q4 Complete each of the following proportions and use it to derive a trigonometric identity:

a. $\dfrac{AD}{AB} =$ b. $\dfrac{BF}{AB} =$

3. Press *Show Relationship 1*. The squares in this figure show a familiar Pythagorean relationship.

Q5 Using lengths of line segments, express relationship 1 as an algebraic equation. Substitute the trigonometric functions into the equation and state the identity. Do the same for relationships 2 and 3.

OTHER ANGLES

It is a little more difficult to make obtuse angles and reflex angles fit in with the right triangle model. The trick is that you have to use directed distances. Normally the length of a line segment must be positive, but this time you will change the rules to suit your needs. Some of the line segments will represent negative numbers.

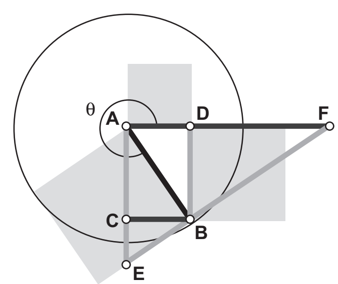
4. Press *Any Angle*. Angle θ is no longer restricted to acute angles.
5. Drag the angle control again. Go all the way around the circle.

Q6 When a line segment changes color, check the corresponding measurement. What is the significance of the color changes?

Q7 Now revisit this proportion from Q3:

$$\frac{BD}{AB} = \frac{BF}{AF}$$

Some of these lengths may now be negative, depending on the angle. Will the proportion still be true for all angles? Check at least one angle in each quadrant.



Q8 Use the action buttons to look at the Pythagorean relationships again. Will any of these relationships change when the signs change? Explain.

Objective: Students review the connections between trigonometric functions and right triangle geometry. They use geometric relationships to justify trigonometric identities.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students should first have a strong understanding of trigonometric function definitions, including the reciprocal functions cosecant, secant, and cotangent.

The activity addresses functions of obtuse angles and reflex angles. If students have only learned functions of acute angles, they should still be able to handle the first section of the activity.

Sketchpad Level: Easy. There are no constructions involved in this activity.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Trigonometric Identities.gsp**) or Whole-Class Presentation (use **Trigonometric Identities Present.gsp**).

ACUTE ANGLES

Q1 There are six triangles similar to $\triangle ABD$. They are $\triangle AFB$, $\triangle EAB$, $\triangle BAC$, $\triangle EFA$, $\triangle EBC$, and $\triangle BFD$. Remind students that it is important to have the vertex labels in the correct order.

Q2 You should check in with students here. If they fail to get this part right, they will have little hope of gaining anything from the rest of the activity.

$BD = \sin\theta$	$AE = \csc\theta$
$AD = \cos\theta$	$BE = \cot\theta$
$AF = \sec\theta$	$BF = \tan\theta$
$AC = \sin\theta$	$BC = \cos\theta$

Q3 Substitute and simplify:

$$\frac{BD}{AB} = \frac{BF}{AF} \Rightarrow \frac{\sin\theta}{1} = \frac{\tan\theta}{\sec\theta} \Rightarrow \sin\theta = \frac{\tan\theta}{\sec\theta}$$

Q4 There are at least two possible answers for each:

a. $$\frac{AD}{AB} = \frac{AB}{AF} \Rightarrow \frac{\cos\theta}{1} = \frac{1}{\sec\theta} \Rightarrow \cos\theta = \frac{1}{\sec\theta}$$

$$\frac{AD}{AB} = \frac{BE}{AE} \Rightarrow \frac{\cos\theta}{1} = \frac{\cot\theta}{\csc\theta} \Rightarrow \cos\theta = \frac{\cot\theta}{\csc\theta}$$

b. $$\frac{BF}{AB} = \frac{BD}{AD} \Rightarrow \frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta} \Rightarrow \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\frac{BF}{AB} = \frac{AB}{BE} \Rightarrow \frac{\tan\theta}{1} = \frac{1}{\cot\theta} \Rightarrow \tan\theta = \frac{1}{\cot\theta}$$

Q5 Relationship 1:

$$BD^2 + AD^2 = AB^2 \Rightarrow \sin^2\theta + \cos^2\theta = 1$$

Relationship 2:

$$AB^2 + BE^2 = AE^2 \Rightarrow 1 + \cot^2\theta = \csc^2\theta$$

Relationship 3:

$$AB^2 + BF^2 = AF^2 \Rightarrow 1 + \tan^2\theta = \sec^2\theta$$

OTHER ANGLES

Q6 The black line segment, AB, always represents the value 1. Blue line segments represent positive values and red segments represent negative values. You have to use both positive and negative numbers because of the association between the line segments and trigonometric functions.

Students might also notice the geometric pattern in the colors. Horizontal segments are positive (blue) to the right and negative (red) to the left. Vertical segments are positive upward and negative downward. The sign of lengths BE and BF is the opposite of the sign of their common slope.

Q7 When you use directed distances, this proportion remains true for any angle for which the line segments are defined. The same is true for all the proportions from Q4.

Q8 All the Pythagorean relationships hold up. In the equations, every term is squared, so the signs of the substituted values do not matter.

1. Open **Trigonometric Identities Present.gsp.** Drag the angle control at the bottom so students can see that it is restricted to acute angles.

The circle has a radius of 1, and you can change the scale using the control below. The vertex of the angle is at the center of a unit circle. Make sure the class understands that all of the linear measurements are based on this unit definition. Segment AB is the terminal side of angle θ. Segment EF is tangent to the circle at point B. All the other line segments are either horizontal or vertical.

Q1 Are there any triangles similar to $\triangle ABD$? ($\triangle AFB$, $\triangle EAB$, $\triangle BAC$, $\triangle EFA$, $\triangle EBC$, and $\triangle BFD$)

Q2 Consider BD/AB. That's the sine of θ, but $AB = 1$. What does that make BD? ($BD = \sin\theta$)

The measure of angle θ is on the screen, so you can confirm these answers by calculating the trigonometric functions of θ.

Q3 In fact, each of these remaining measurements corresponds to one of the trigonometric functions. Which is which? Work this out together and keep track of it on the board or in the sketch:

$BD = \sin\theta$	$AE = \csc\theta$
$AD = \cos\theta$	$BE = \cot\theta$
$AF = \sec\theta$	$BF = \tan\theta$
$AC = \sin\theta$	$BC = \cos\theta$

Q4 The similar triangles give us a lot of proportions. Here's one: $BF/AB = BD/AD$. Substitute the trigonometric functions from Q3 for those line segment lengths. What do you get? ($\tan\theta = \sin\theta/\cos\theta$)

Q5 Interpret several other proportions in the same way.

Q6 Press *Show Relationship 1.* By the Pythagorean theorem, $BD^2 + AD^2 = AB^2$. What trigonometric identity follows from that? ($\sin^2\theta + \cos^2\theta = 1$) Do the same with the other Show buttons:

Relationship 2: $AB^2 + BE^2 = AE^2$, so $1 + \cot^2\theta = \csc^2\theta$

Relationship 3: $AB^2 + BF^2 = AF^2$, so $1 + \tan^2\theta = \sec^2\theta$

2. Press *Any Angle.* Now angle θ is no longer restricted to acute angles. Drag the angle control to show this. The measurements are actually directed distances. Their signs change with the corresponding functions, and the line segment colors change to emphasize this.

3. Examine the same identities again. Have the class discuss the question of whether the identities continue to hold for obtuse angles and reflex angles.

Law of Sines

You have already used the sine, cosine, and tangent ratios to find missing parts of triangles. However, the definitions of these functions (involving the ratios of the opposite, adjacent, and hypotenuse) apply only to right triangles. In this activity you'll explore a different set of ratios that you can use in oblique triangles.

MORE RATIOS

1. In a new sketch, construct a triangle. Measure each angle and each side.

To show or change a label, select the object and then choose **Display | Label.**

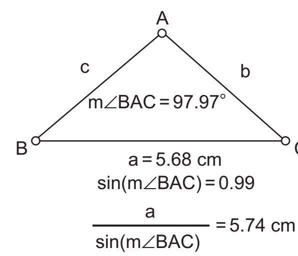
2. The vertices are automatically labeled *A, B,* and *C*. Label the sides *a, b,* and *c* according to the vertex that is opposite each side.

Q1 Drag the vertices to make $\angle A$ larger than $\angle B$. Which side is longer, *a* or *b*? Is this always true? Write down the measurements from three different examples.

Q2 Drag the vertices to make side *c* longer than *b*. Which angle is larger, $\angle B$ or $\angle C$?

To calculate the sine, choose **Measure | Calculate.** Select **sin** from the Function pop-up menu, and then click the angle measurement in the sketch.

3. Calculate the sine of each angle.



4. Calculate the ratio of the length of each side to the sine of the opposite angle.
5. Select all three ratios and place them in a table by choosing **Graph | Tabulate.** With the table still selected, choose **Graph | Add Table Data** and choose to add ten entries as the values change.

Q3 Drag the vertices to change the angles and side lengths. What do you observe about the ratios?

Q4 Write your observation as an equation.

Q5 Calculate the reciprocal of each ratio. What do you observe? Write an equation.

These equations are both ways of writing the Law of Sines.

EXPLORE MORE

Q6 Open **Law of Sines Proof.gsp.** Use the labels in the blue triangle to write a formula for sin *A*. Use the labels in the pink triangle to write a formula for sin *B*.

Q7 The length of segment *h* appears in both formulas. Solve both formulas for *h*, and set the results equal to each other.

Q8 What must you do to this equation to complete a proof of the Law of Sines?

Q9 By dragging point *C*, you can move segment *h* so it's outside $\triangle ABC$. Is your proof of the Law of Sines still correct, or must you modify it? Explain.

Objective: Students construct a triangle, measure some ratios, and find that certain ratios remain equal no matter the shape of the triangle: the Law of Sines. They then use a prepared sketch and apply the triangle definition of the sine function to develop a proof of the Law of Sines.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students must be familiar with the Pythagorean theorem and with the definition of the sine function in a right triangle.

Sketchpad Level: Intermediate. Students must use the Calculator and measure angles and segment lengths.

Activity Time: 15–25 minutes

Setting: Paired/Individual Activity (start with a blank sketch; use **Law of Sines Proof.gsp** for the Explore More section) or Whole-Class Presentation (use **Law of Sines Present.gsp**)

The activity itself does not address the ambiguous case, nor does it provide practice problems for applying the Law of Sines. The Whole-Class Presentation provides both. For this reason you should strongly consider using the presentation sketch as a follow-up to the activity.

MORE RATIOS

Q1 When $\angle A$ is larger than $\angle B$, side a is always longer than side b. Students should record three different sets of all four values ($\angle A$, $\angle B$, a, and b) that they tried.

Q2 When side c is longer than b, $\angle C$ is larger than $\angle B$.

Q3 The ratios remain equal to each other despite changes in the triangle.

Q4 In equation form this is the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Q5 The reciprocals of equal nonzero values must also be equal. The second form of the Law of Sines is

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

EXPLORE MORE

Q6 By construction, the pink and the blue triangles are always right triangles.

$$\sin A = \frac{h}{b} \qquad \sin B = \frac{h}{a}$$

Q7 Solving for h, the two equations are $h = b \cdot \sin A$ and $h = a \cdot \sin B$. Setting the right-hand sides equal to each other gives $b \cdot \sin A = a \cdot \sin B$.

Q8 Dividing this result by ab gives

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

Alternatively, dividing the result by $\sin A \cdot \sin B$ gives

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

Both results are correct statements of the Law of Sines. Once students have written the Law of Sines as the equality of two ratios, ask them to justify adding the third ratio (involving c and $\sin C$) to the equation.

Q9 Once you drag C so h is outside the triangle, one of the two right triangles involves an exterior angle of the triangle. For instance, if you drag C past B, the right triangle DBC no longer contains $\angle ABC$, but now contains the exterior angle at B, which is supplementary to $\angle ABC$. Because $\sin B = \sin(180° - B)$, the Law of Sines is still correct.

WHOLE-CLASS PRESENTATION

Use **Law of Sines Present.gsp** to present this activity to the entire class. Even when students have done the activity on their own computers, use page 4 to demonstrate the ambiguous case.

Page 1 (Intro) describes the contents of the sketch.

Page 2 (Ratios) parallels the main part of the student activity. Use the buttons and directions in the sketch to present this part.

Page 3 (Proof) parallels the Explore More section of the activity.

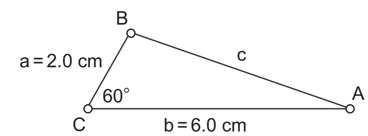
Page 4 (Ambiguous) explores the ambiguous SSA case.

Pages 5 and 6 (SSA and AAS) make it easy to generate practice problems and their solutions.

Law of Cosines

You know how to use sine, cosine, and tangent functions to solve right triangle problems, and you know how to use the Law of Sines to solve some problems in triangles without right angles. These two methods are very important, but they are not enough to solve all triangle problems. In this activity you'll find another law that allows you to solve problems for which neither of the other methods works.

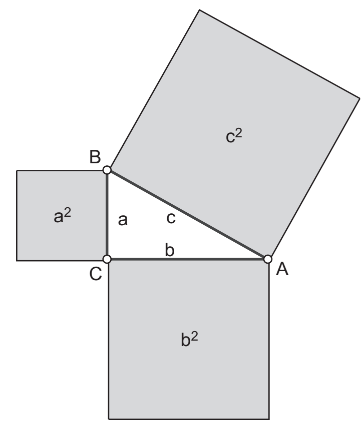
TRY EXISTING METHODS



Q1 Why can't you use the sine, cosine, or tangent functions directly to find the missing measurements in this triangle?

Q2 Write down the Law of Sines and substitute into it the measurements from this triangle. Can you use this equation to solve for the missing side or angles? Explain.

Q3 Can you use the Pythagorean theorem to find the missing side? Give a reason.



The figure at right illustrates the Pythagorean theorem. If you square each side, then $c^2 = a^2 + b^2$. Unfortunately, you can use this theorem only if one of the angles is a right angle. Even so, the Pythagorean theorem is a useful starting place.

EXTEND THE PYTHAGOREAN THEOREM

1. Open **Law of Cosines.gsp.** Measure the lengths of all three sides of the triangle. Also measure $\angle C$.

Choose **Measure | Calculate.** To enter a measurement into your calculation, click it in the sketch.

2. Calculate the value of a^2 (the area of the square on a). Also calculate b^2 and c^2.

Q4 Calculate $a^2 + b^2 - c^2$. What result do you get? Why?

3. Drag points A and B.

Q5 Why do two of the lengths remain constant? What length does change?

4. You don't need the circles any more, so hide them.

Q6 For what values of $\angle C$ is $a^2 + b^2 - c^2$ positive? For what values is it negative?

GRAPH

It will be useful to figure out exactly how the value of $a^2 + b^2 - c^2$ depends on $\angle C$. To investigate, you'll graph $a^2 + b^2 - c^2$ as a function of $\angle C$.

5. Press *Show Axes.* To plot the current values of the quantities you're investigating, select $m\angle C$ and $a^2 + b^2 - c^2$ in order, and choose **Graph | Plot As (x, y).**

6. Turn on tracing for the plotted point. Then vary $\angle C$ by dragging A and B.

To trace the point, select it and choose **Display | Trace Plotted Point.**

Q7 Have you graphed a function with a similar shape before? What function?

7. Use $f(x) = \cos(x)$ as the parent function. Plot this function.

Choose **Graph | Plot New Function.** Choose **cos** from the Function pop-up menu. Then click *x* on the keypad.

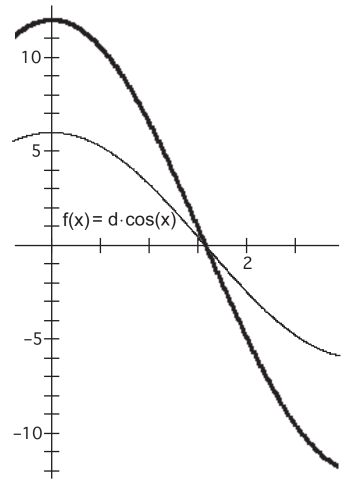
Q8 How is the graph of $f(x) = \cos(x)$ similar to the trace, and how is it different?

Q9 What transformation could you use to make the function match the trace?

8. Create parameter d and set its value to 1.

To create a parameter, choose **Graph | New Parameter** and use the dialog box that appears to set the name and value of the parameter.

9. Edit function $f(x)$ by double-clicking it. Change it to $f(x) = d \cdot \cos(x)$.



Q10 Change the value of d by selecting it and pressing the + key or − key repeatedly. How does d affect the graph?

Q11 Adjust d until the graph matches the trace. What value of d did you use to do this?

When you studied stretches and shrinks, you probably used *a* and *b* as parameters. They are called *d* and *e* here to avoid confusion with the sides of the triangle.

10. Record the values of a, b, and d needed to match the two functions by selecting all three values and choosing **Graph | Tabulate.** Double-click the table to make the first row of values permanent.

11. Use the sliders to change lengths a and b. Then erase the traces and make a new trace. Adjust d to match the function to the new trace, and then record the values again by double-clicking the table. Repeat this step until you have five rows in the table.

If you record a row of values incorrectly, you can remove it by selecting the table and choosing **Graph | Remove Table Data.**

Q12 Examine your table. How can you express d in terms of a and b? Substitute this for d in the function definition. What is the new definition?

Q13 The function is equal to $a^2 + b^2 - c^2$. Solve the resulting equation for c^2. This is the Law of Cosines.

EXPLORE MORE

Q14 What happens to the Law of Cosines if $\angle C$ is a right angle? Substitute the appropriate value and simplify the result. What do you end up with?

Q15 You can also use the Law of Cosines to find an angle when you have all three sides. In that case, how would you write the equation?

Objective: Students develop the Law of Cosines by exploring numerically and graphically how the Pythagorean theorem fails for triangles without a right angle. By transforming the cosine function, they find an expression for the discrepancy between c^2 and $a^2 + b^2$. This expression is the term needed to complete the Law of Cosines.

Student Audience: Algebra 2/Precalculus

Prerequisites: Students must be familiar with the Pythagorean theorem, with the graphs of the trigonometric functions, and with stretching and shrinking functions.

Sketchpad Level: Intermediate. Students perform a number of measurements and calculations, plot a point and a function, and tabulate data.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Law of Cosines.gsp**) or Whole-Class Presentation (use **Law of Cosines Present.gsp**)

Begin by telling students that they will work out a new way to find missing information in triangles by using the Pythagorean theorem, the graphs of the trigonometric functions, and function transformations. You may want to review some of those topics prior to using this activity.

TRY EXISTING METHODS

Q1 You can't use the sine, cosine, or tangent ratios directly because they apply to right triangles, and this triangle is not a right triangle.

Q2 The Law of Sines is insufficient because one value for each of the three ratios is unknown.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\frac{\sin A}{2} = \frac{\sin B}{6} = \frac{\sin 60°}{c}$$

Q3 You cannot use the Pythagorean theorem because this triangle is not a right triangle.

EXTEND THE PYTHAGOREAN THEOREM

Q4 The value of $a^2 + b^2 - c^2$ is zero at first because this triangle is initially a right triangle, and so the Pythagorean theorem applies to it.

Q5 The lengths of sides a and b stay constant because points A and B are attached to circles which are centered on point C. Side c is the only one that changes.

By keeping a and b constant, students can concentrate on the relationship between $\angle C$ and the quantity $a^2 + b^2 - c^2$. This relationship will reveal the connection with the cosine.

Q6 If $m\angle C < 90°$, the value is positive. If $m\angle C > 90°$, the value is negative.

GRAPH

Q7 The point traces a shape similar to a cosine graph, with a maximum at 0° and a minimum at 180°.

Q8 The graph has the same period as the trace, but is much flatter. The graph's maximum value is 1, but the trace's maximum value is 12.

Q9 To match the trace, you need to stretch the function vertically, increasing its amplitude by a factor of 12.

Q10 Changing d results in a vertical stretch or shrink. To try smaller increments of d, students can double-click the parameter, or they can change its properties to use a keyboard adjustment value of 0.1.

Q11 To match the function to the trace, d must be 12.

Students collect five rows of data. Be sure that they erase traces between trials and that they accurately match the trace and the function plot before recording data.

Q12 Based on the table, $d = 2 \cdot a \cdot b$. The new function is $f(x) = 2ab\cos x$.

Q13 Solving for c^2:

$$2ab\cos C = a^2 + b^2 - c^2$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

EXPLORE MORE

Q14 Because $\cos(90°) = 0$, it simplifies to $c^2 = a^2 + b^2$.

Q15 To find $\angle C$, given a, b, and c:

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

You can use this presentation sketch to gather and analyze data from a triangle in order to discover the Law of Cosines.

1. Open **Law of Cosines Present.gsp.** On page 1, begin with three questions.

Q1 Why can't you use the sine, cosine, or tangent functions to find the missing measurements in this triangle? (The triangle is not a right triangle.)

Q2 Write down the Law of Sines and substitute into it the measurements from this triangle. Can you use this equation to solve for the missing side or angles? Explain. (No, each ratio in the Law of Sines still contains an unknown quantity.)

Q3 Can you use the Pythagorean theorem to find the missing side? Give a reason. (No, the Pythagorean theorem works only for right triangles.)

This activity is devoted to finding a new method that does work for triangles like this.

2. Go to page 2 and tell students that they'll begin by calculating a quantity related to the Pythagorean theorem. Use the buttons to measure the sides and to measure $\angle C$.

We use this name because the expression is the difference between the two sides of the Pythagorean theorem.

3. Calculate $a^2 + b^2 - c^2$, the *Pythagorean difference.*

Q4 Why is this value 0 at first? (The triangle starts with a right angle.)

Q5 Drag A or B to observe how the Pythagorean difference changes as $\angle C$ changes. When is it positive? When is it negative? When is it zero?

4. Plot the Pythagorean difference as a function of $\angle C$. Animate A to fill it in.

Q6 What function does this remind you of? (the cosine function)

5. On page 3, show the plot of the Pythagorean difference. Then show the cosine function.

Q7 The cosine function is too flat. How can you transform it to match the other plot? (with a vertical stretch)

6. Show the transformed function, and adjust d until the curves match.
7. Show the table and double-click it to record the current values. Then change a and b, match again, and record more values. Repeat.

Q8 How is d related to a and b? ($d = 2ab$) Go to page 4, show the table there, and ask if the relationship holds.

8. On page 4, substitute and solve in order to arrive at the Law of Cosines.
9. Finish by asking students to write the two other forms of the Law of Cosines by using $\angle A$ or $\angle B$ in place of $\angle C$.