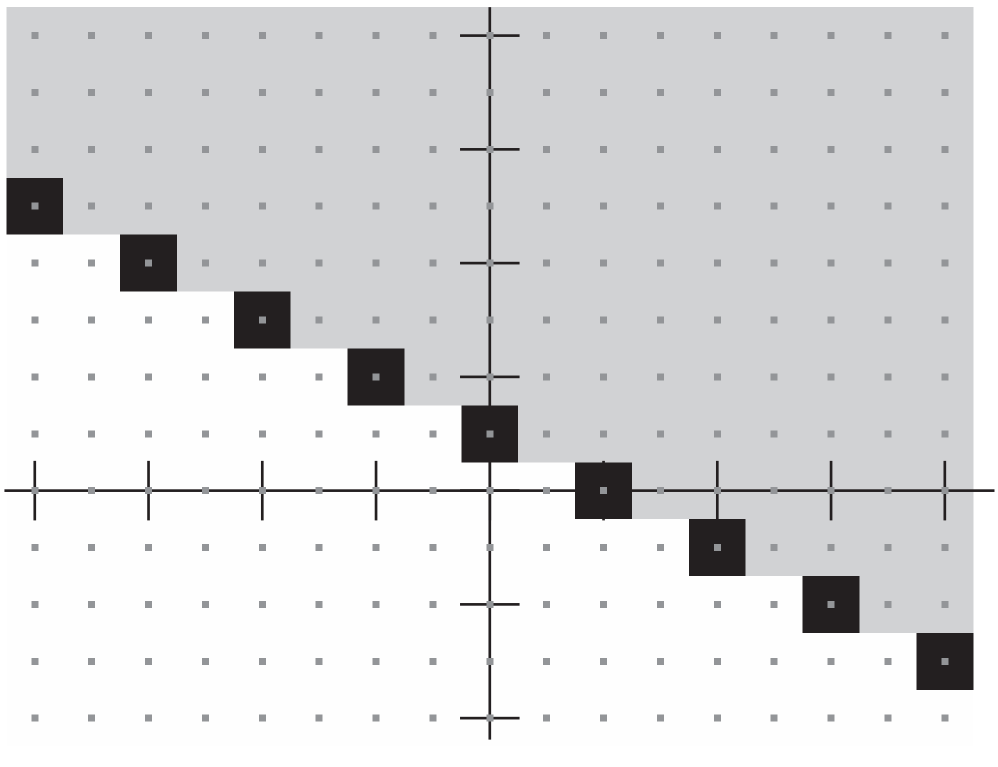
3

Systems



Solving Systems of Equations

The graphical solution to a system of two equations is the intersection of the graphs of the equations. In real life, you can often solve a system of equations to make a calculated choice between two options. In this activity you'll solve a system of equations to make some decisions for a truck rental company.

SKETCH

The end of the month is a popular time to move, so many people rent moving trucks. Each truck rental company has its own formula to determine the price of renting a truck. The rental rates depend on two factors: the *Drive-off Fee* and the *Price per Mile Driven*. Use the table below to compare the rates of two different companies.

Rate Comparison for 14-foot Moving Trucks

Company	Drive-off Fee	Price per Mile Driven
Cercano Rentals	$12.00	$2.75
We-Haul-It	$42.50	$0.40

With the rate information, the renter can make an informed decision. The price of renting a 14-foot truck will depend on the number of miles driven. Here is the price formula for Cercano Rentals:

$$f(x) = 12.00 + 2.75x$$

Q1 What is the price formula for We-Haul-It rentals?

Which company is cheaper depends on how many miles you drive. To compare the prices, you can graph each equation and find the intersection of the graphs.

The sketch is blank, but this document contains some tools that will be useful to you.

1. Open **Solving Systems.gsp** and choose **Graph | Define Coordinate System.** Hide the point at the origin and the point at (1, 0). Use the **Arrow** tool to drag one of the numbers along the x-axis closer to the origin until you see "20, 40, 60, …" for the scale on the x-axis. Then drag either axis to move the origin to the lower left-hand corner of the screen.

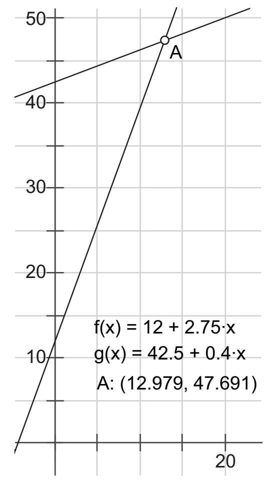
You can adjust the window, the scroll bars, and the scale of the x-axis to improve your view of the graph.

2. Choose **Graph | Plot New Function.** Type 12.00 + 2.75*x and click OK. The graph and its equation appear on the screen. With the graph and equation selected, choose **Display | Color** and make them blue.
3. Make sure no objects are selected. Choose **Graph | Plot New Function.** Type 42.50 + 0.40*x and click OK. The graph and its equation appear on the screen. Use **Display | Color** to make them dark green.

Solve this system of equations by finding the intersection of the graphs. First find an approximate solution.

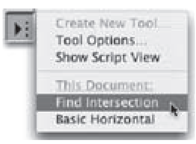
4. Click the **Point** tool on one of the graphs. A point appears on the graph.

5. Use the **Arrow** tool to drag the point to the approximate intersection of the two lines. Choose **Measure | Coordinates.**



Q2 What are the values and units of the *x*- and *y*-coordinates that appear?

6. Now find a more exact location of the intersection. Drag the point away from the intersection to avoid future confusion.



7. Press and hold the **Custom** tools icon. From the menu that appears, choose the **Find Intersection** tool.

8. Click this tool on the two functions $f(x)$ and $g(x)$ in the sketch. (Be sure to click the functions themselves, not the function plots.) Then click a spot on the *x*-axis to the left of the intersection point, and finally a spot on the *x*-axis to the right of the intersection point.

The tool constructs an intersection of the two functions between the *x*-values of the two points and shows the coordinates of the intersection point.

Q3 What are the coordinates of the point of intersection? What are the units for each coordinate? What is the significance of this point?

Q4 Why do you think the point of intersection is often called the *break-even point*?

Q5 If the renter of the moving truck will drive 10 miles, from which company should he rent?

Q6 In Spanish, *cercano* means "nearby." Do you think that the name Cercano Rentals is appropriate for the way people use its trucks?

EXPLORE VARIATIONS

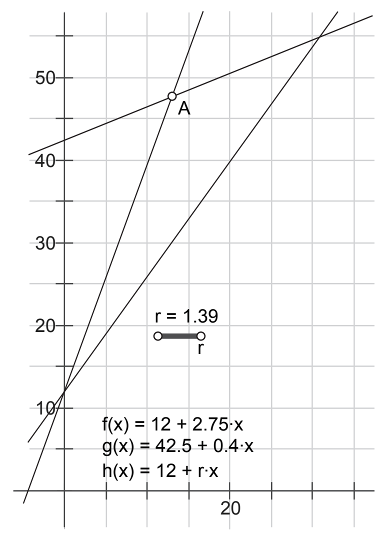
The owners of Cercano Rentals would like the company to be more competitive in moves of longer distances. The company investigates how it can change its rates to attract more long-distance movers. For marketing reasons, the company believes it is important to maintain its low *Drive-off Fee* of $12.00. Therefore, it decides to explore different scenarios by adjusting its *Price per Mile Driven*. Create a slider to make the *Price per Mile Driven* a variable rate.

9. Choose the **Basic Horizontal** tool and click in a blank space in the sketch to construct a slider.

To change the measurement's label, double-click it with the **Text** tool.

10. Change the label of the slider's measurement to *r*. Also change the label of the slider's active endpoint to match. The label *r* will represent the varying *Price per Mile Driven*. Drag the slider's endpoint back and forth to see how the measurement behaves.

11. Choose **Graph | Plot New Function.** Type 12.00 + r*x and then OK. (Click *r* in the sketch to insert it into the formula.). The graph and its equation appear on the screen. Choose **Display | Color** and make the line red.



As you drag the point at the end of the slider to the left or right, you will notice that the slope of the red line changes and, as a result, so does the break-even point.

Use the **Find Intersection** tool to get accurate results to these questions.

Q7 Approximately what *Price per Mile Driven* should Cercano Rentals charge so that it is more competitive for moves of less than 40 miles? Less than 50 miles? Less than 60 miles? Less than 100 miles?

AN INTERNATIONAL QUANDARY

Add a page to the sketch by choosing **File | Document Options.**

Diego recently moved to California from Argentina and enjoys calling his family and friends back home to stay in touch. Currently, Diego pays \$0.69 per minute for his calls to Argentina. The local phone company is encouraging Diego to buy an international calling plan. If he buys the plan, he will pay \$3.99 per month plus an additional \$0.15 per minute for his calls to Argentina.

Q8 On a new page, create and solve the system of equations to determine the minimum number of minutes of phone calls per month for which it would make sense for Diego to buy the international calling plan. Write down the equations you used and the break-even point.

EXPLORE MORE

Q9 If one or more of the equations in the system are non-linear, is there a solution to the system? Can there be more than one intersection point?

Q10 Can you write a system of equations that has two solutions? If yes, try it out.

Objective: Students find the intersection of two graphs to solve a system of equations that reflects a real-world scenario. They then use a slider to see how the solution changes as the situation changes.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should understand how to graph linear equations. It's best if they've already been introduced to systems of equations.

Sketchpad Level: Intermediate. Students plot functions on the graph and use two custom tools.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (use **Solving Systems.gsp**) or Whole-Class Presentation (use **Solving Systems Present.gsp**)

SKETCH

Q1 The price formula for We-Haul-It is

$$g(x) = 42.50 + 0.40x$$

Q2 Answers will vary for the values, but should be close to (13, 47.7). The units are miles and dollars, so students might express the result as (13 miles, \$47.70). Remind students that the y-value should be rounded to two decimal places because it is currency.

Q3 The numerical values to three decimal places are (12.979, 47.691). With the units, and rounding off the y-value to the nearest cent, the solution can be written (12.979 miles, \$47.69). This point represents the distance at which the two rental companies have exactly the same cost.

Q4 It is called the *break-even point* because it is the place where the price to rent a truck is the same for both companies. The renter will neither lose nor save money by using the other company.

Q5 He should rent from Cercano Rentals because it will be cheaper.

Q6 Yes, Cercano Rentals is good for moves that are nearby, because it is cheaper for moves of less than about 13 miles.

Q7 The technique of using a slider to answer what-if questions is a powerful one. It's a good idea to talk about this technique during the wrap-up discussion.

To match We-Haul-It for a 40-mile trip, Cercano needs to set its price per mile at \$1.16. Here are the rates Cercano must use to match We-Haul-It at other distances:

50 mi: \$1.01 60 mi: \$0.91 100 mi: \$0.70

Note: Student answers may vary by a few cents because of limitations in manipulating the slider.

In case the intersection goes off-screen, remind students that they can rescale the axes by dragging the tick numbers.

Q8 This is the system of equations:
Current phone plan: $f(x) = 0.69x$
Calling card plan: $g(x) = 3.99 + 0.15x$
The point of intersection is (7.389 minutes, \$5.09). If Diego talks for more than about seven minutes a month, the international calling plan will save him money.

Q9 Yes, there can still be a solution to a system of equations even if not all of the equations are linear.

Yes, there can be more than one intersection point.

Q10 Answers will vary. A simple example to show is a parabola with any linear equation that intersects it. For example, the system $f(x) = x^2 - 6x + 9$ and $g(x) = 0.5x + 1$ has two solutions. If you plot these two functions, you can use the **Find Intersection** tool to determine the actual intersections.

VARIATIONS

You may wish to give students different types of real-life word problems that involve finding an intersection. You could also ask them to make up their own break-even point problem.

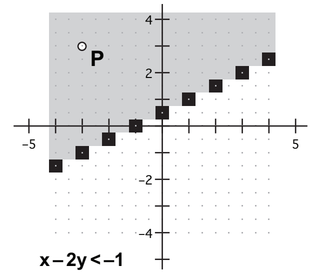
WHOLE-CLASS PRESENTATION

You can present this activity to the entire class using **Solving Systems Present.gsp.**

Use the buttons and directions there to guide the presentation, and be sure to solicit student ideas and encourage discussion as you present.

Graphing Inequalities in Two Variables

Imagine yourself as a point in the coordinate plane, free to wander. In this activity you'll travel to different locations, learning how to keep inequalities satisfied.



KEEP ME SATISFIED

1. Open **Graphing Inequalities.gsp.** Move point P around the plane, and observe how the coordinates of the point vary.

Q1 Where can you move the point so that the calculation $2x + 3$ stays greater than 5? Describe these locations.

To edit the calculation, double-click it. Click the y measurement in the sketch to enter it into the calculation.

Q2 Edit the calculation to $2y + 3$. Now where can you move the point so that the calculation of $2y + 3$ stays greater than 5? Describe these locations.

Q3 Press *Show Tracer* and move point P around again, this time anywhere you like. What happens to the tracer when the calculation $2y + 3$ is greater than 5, when it's equal to 5, and when it's less than 5?

Although the inequality $2y + 3 > 5$ has only one variable, we're still interested in all the coordinate points, which have two variables.

A graph of an inequality in two variables is the collection of all the points in the plane that *satisfy* the inequality. In this context, satisfy means "make true." For example, any time point P is in a position where $2y + 3$ is greater than 5, the inequality $2y + 3 > 5$ is true.

Q4 What points satisfy the equation $2y + 3 = 5$? How might knowing this help you describe the points that satisfy the inequality $2y + 3 > 5$?

EXPLORE

To change the calculation, double-click it and then use the Calculator. Enter values of x or y by clicking the x or y measurements in the sketch.

2. Erase the traces, hide the tracer, and change the calculation to $3x - 2y$.

Q5 Given any position of point P, if you move it straight down (without moving it left or right), the value of $3x - 2y$ increases. Explain why. What happens to the calculation if you move P to the right without moving it up or down? Why?

3. Show the tracer and move point P around the plane.

Q6 Describe the locations of point P where $3x - 2y > 5$ is true. Use your answer to Q4 to explain why any point to the right of (or below) the line $3x - 2y = 5$ is a point that satisfies the inequality $3x - 2y > 5$.

4. On page 2, drag point P and notice how it moves between closely spaced points. Show the tracer and drag P again. The tracer changes color, showing for each location of point P whether $x + 2y > -1$ is true.

Q7 Use your answers to Q5 and Q6 to explain why one side of the line $x + 2y = -1$ contains *all* the points where $x + 2y > -1$.

Q8 Without using Sketchpad, use your answers to Q4 through Q6 to predict what points in the plane will satisfy the inequality $3x - y > 2$. Write out your thinking in a sentence, and draw a sketch of the graph on your paper. Then check your answer by editing the calculation and moving point P.

Q9 For each inequality below, sketch on your paper your prediction of the graph. Then use Sketchpad to check your result.

a. $-x + 2y > -1$

b. $x + 2y > 1$

c. $-x + 2y > 1$

d. $x - 2y > 1$

Q10 Use page 3 to find points that satisfy the inequality $y < x^2$. How can you determine, without Sketchpad, which points in the plane satisfy $y < x^2$?

Q11 Answer Q10 for the inequality $y > 3x - 5$.

EXPLORE MORE

To see the entire graph quickly, select both P and the tracer square, and choose **Construct | Locus.** Then use **Edit | Properties | Plot** to increase the number of samples so that the locus fills the entire region.

Q12 Use page 4 to find the points in the plane that satisfy the inequalities below. Draw a sketch of each solution. Think about how you could go about predicting the points that satisfy the inequality without Sketchpad available.

a. $x^3 + 4 > 12$

b. $x^3 + 4 < 12$

c. $x^3 + 4 < -12$

d. $|x| - |y| < 3$

e. $|x| - |y| < -3$

f. $|x| - |y| > 3$

Q13 When you're done, compare the solutions to a, b, and c with each other, and the solutions to d, e, and f with each other. What conclusions, if any, can you draw about reversing the inequality sign?

Q14 Use page 5 to find the points that satisfy the inequalities below. How could you predict the points that satisfy the inequality without Sketchpad available?

a. $(x - 2)(x + 1) < 0$

b. $(x - 2)(x + 1)(x - 4) < 0$

Q15 Use page 6 to find points that satisfy the inequality $3x - xy^2 + y^3 < 5$. (Make sure you drag point P to a variety of locations.) What points would satisfy the inequality $3x - xy^2 + y^3 > 5$? Why? What points would satisfy the inequality $-(3x - xy^2 + y^3) < -5$? Why? Check your answers with the sketch and explain the results that you see.

Objective: Students explore the graphs of inequalities by dragging a point on the Cartesian plane and observing where the inequality is true and where it is false. Students investigate the relationship between an inequality and the corresponding equation, and compare the graphs of different (but related) inequalities.

Student Audience: Algebra 1/Algebra 2

Prerequisites: None

Sketchpad Level: Easy

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Graphing Inequalities.gsp**) or Whole-Class Presentation (use **Graphing Inequalities Present.gsp**)

Related Activities: Graphing Systems of Inequalities

KEEP ME SATISFIED

Q1 The calculation $2x + 3$ stays greater than 5 as long as P is to the right of a vertical line through 1 on the x-axis.

Q2 The calculation $2y + 3$ stays greater than 5 as long as P is above a horizontal line through 1 on the y-axis.

Q3 The tracer is cyan (light blue) when the calculation is greater than 5, black when it's equal to 5, and magenta (light purple) when it's less than 5.

Q4 The graph of $2y + 3 = 5$ is the horizontal line with a y-intercept of 1. The points satisfying $2y + 3 > 5$ are all points above that line.

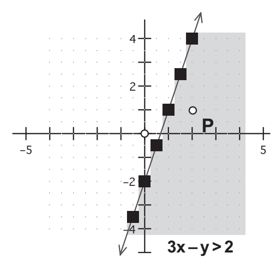
EXPLORE

Q5 When you move P straight down, the y-value decreases. In the expression, y is multiplied by -2, so the value of the $-2y$ term increases when y decreases. Similarly, when you move P to the right, the x-value increases, so the value of the $3x$ term also increases.

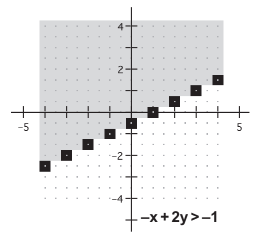
Q6 The statement $3x - 2y > 5$ is true when P is to the right of or below the line that is the graph of $3x - 2y = 5$. Because the value of the expression is 5 when P is on the line, and the value increases when you move P to the right or down, every point on this side of the line must correspond to an expression value greater than 5.

Q7 In the expression $x + 2y$, the value of the expression increases when you move P to the right or up, because the first term (x) increases when you increase x by moving P right, and the second term ($2y$) increases when you increase y by moving P up. For this reason, the statement $x + 2y > -1$ is true when P is to the right of or above the line that is the graph of $x + 2y = -1$.

Q8 To satisfy the inequality $3x - y > 2$, the value of the expression must be greater than the value of the expression on the line $3x - y = 2$. The x term has a positive coefficient, so P must be to the right of the line; the y term has a negative coefficient, so P must be below the line. All points to the right of and below the line satisfy the inequality.



Q9 a.



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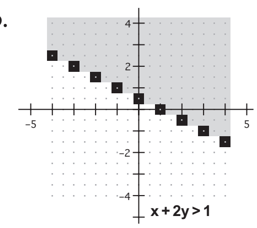
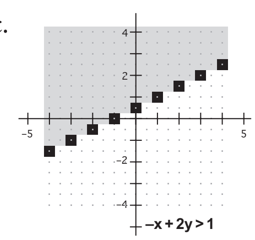
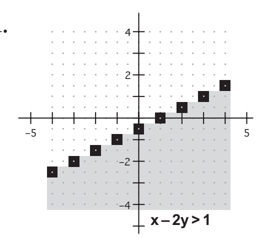


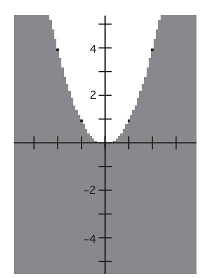
c.



d.



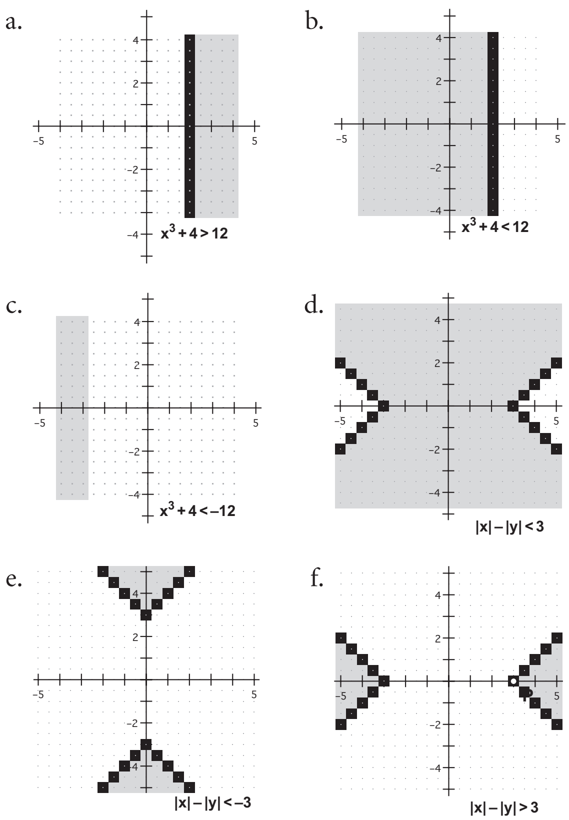
Q10 The points that satisfy the inequality $y < x^2$ are the points below the parabola $y = x^2$.



Q11 The inequality $y > 3x - 5$ is true when y is above the line because y has a positive coefficient and is on the greater-than side of the inequality. It's true when x is left of the line because x has a positive coefficient and is on the less-than side of the inequality.

EXPLORE MORE

Q12

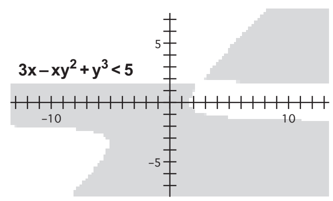


Q13 After students sketch the graphs, comparing graphs a and b and graphs d and f helps them to see that the direction of the inequality sign determines whether the cyan region or the magenta region represents the solution.

The curve(s) representing equality divide the plane into two regions. If you reverse the sign of the inequality, you change which region is the solution.

Q14 Each of these inequalities has a factored polynomial on the left side and zero on the right. To determine whether the inequality is satisfied for a given point, you only need to know the sign of the left side. Find that by determining the sign of each factor. The boundaries are vertical lines at the roots. Every time you cross one, one of the factors changes sign, and that changes the sign of the whole polynomial. Watch out for double roots; none are used in these examples.

Q15 This inequality has an interesting shape. Students may want to adjust the grid extent to view more of it.



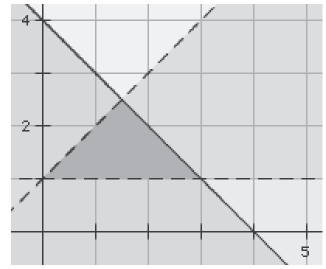
The related inequality $3x - xy^2 + y^3 > 5$ has the inequality sign reversed, so the solution is the complementary region (the region not colored in the diagram). The inequality $-(3x - xy^2 + y^3) < -5$ has the same shape as $3x - xy^2 + y^3 > 5$, because the direction of the inequality is reversed, and the signs of both sides have also been reversed.

If you change the inequality to an equation and solve for x in terms of y, you can graph the result. (In the Function Calculator, set the Equation form to be $x = f(y)$.) You'll need to increase the number of samples in the function plot to see the complete graph.

WHOLE-CLASS PRESENTATION

Use the sketch **Graphing Inequalities Present.gsp** to present this activity to the class. The presentation follows the steps and questions of the student activity, but is streamlined to make it easy to present. Each page of the presentation sketch has directions and buttons to help you present the activity. Use pages 1 through 7 to present the main part of the activity. Pages 8 through 10 correspond to the Explore More questions.

Graphing Systems of Inequalities



When you solve a system of equations graphically, you usually get a single point as the solution. In this activity you'll solve a system of inequalities graphically, and see what the solution of such a system looks like.

A SINGLE INEQUALITY

Begin by graphing the equation $y = \frac{3}{2}x - 2$ and the related inequality $y < \frac{3}{2}x - 2$.

1. In a new sketch, graph $y = \frac{3}{2}x - 2$ by choosing **Graph | Plot New Function** and entering the equation into the New Function dialog box.

Q1 What is the slope of the line and what are its x- and y-intercepts?

Q2 Based on the graph of the *equation,* what do you think the graph of the *inequality* will look like?

To delete the graph, select it and press the Delete or Backspace key on your keyboard.

2. Delete the function plot you just created, but leave the function itself; you will use it again in a moment.

To graph the inequality, you will use a custom tool from **Inequality Tools.gsp.**

3. Open **Inequality Tools.gsp.** Then switch back to your original sketch. (As long as **Inequality Tools.gsp** is open, you can use the tools it contains.)

Switch back to the **Arrow** tool after the graph appears.

4. Press and hold the **Custom** tools icon, and choose **Inequality Tools | y < f(x)** from the menu that appears. Click the function $f(x)$ in your sketch. A graph of the inequality appears.

You can also right-click (Windows) or control-click (Mac) on the strips to get to the Properties dialog box.

5. If the graph of the inequality appears as strips, you can change the Plot Properties of this object so that it appears as a solid area. Select the strips, choose **Edit | Properties,** and use the Plot panel of the Properties dialog box to increase the number of samples. (Approximately 400 should be enough.)

Q3 Is the boundary of the inequality graph solid or dashed? Explain why its appearance makes sense in terms of the inequality.

SYSTEMS OF INEQUALITIES

Next you'll graph a second inequality ($x \geq -y + 1$) on the same coordinate system.

This function should be labeled *g*. If not, choose **Display | Label Function** to change its label to *g*.

6. Create another function by choosing **Graph | New Function.** This inequality expresses x in terms of y, so use the Calculator's Equation pop-up menu to choose **x = f(y).** Then enter the equation $g(x) = -y + 1$.

Q4 What do you think the graph of $x \geq -y + 1$ will look like? Write down your guess before you construct the graph.

Remember to switch back to the **Arrow** tool once the graph appears.

7. To create the graph, choose the custom tool **Inequality Tools | x >= f(y).** Click the tool on the function $g(y)$ in your sketch.

Q5 Which shaded area contains points that satisfy both inequalities? Construct a point in that area, measure its coordinates, and use algebra to confirm that the point satisfies both inequalities.

In addition to the colors in the color menu, you can choose **Other…** to specify a different shade or tint.

Q6 For each system of inequalities below, add a new page to your document (using **File | Document Options**) and construct a graph of the system. If the graphs of two inequalities appear in the same color, change the color of one of the graphs so you can easily see the area of overlap.

For problem h, use **Edit | Preferences | Units** and change the angle units to radians.

a. $y \geq 2x - 1, y < -\frac{1}{2}x + 3$

b. $y < 2, x \geq -3$

c. $y \geq \frac{1}{2}x^2, x < y$

d. $y < \frac{1}{2}x^2, x \geq y - 1$

e. $x < y - 3, y \leq x - 1$

f. $y > \frac{x}{10} - x^2 + 3, y < 2$

g. $x \leq \sqrt{9 - y^2}, y > 1, x \geq -1$

h. $y < \sin x, y \geq 0.5$

EXPLORE MORE

Q7 Change these inequalities so that each variable appears on only one side of the inequality, making sure to keep the properties of inequality in mind. Then graph the system and describe the solution.

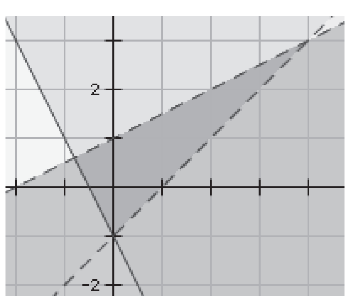
a. $2y - x < 2, |2x| \geq 2y - x$

b. $y^2 \leq x + 2y - 2, x - 3 < 0$

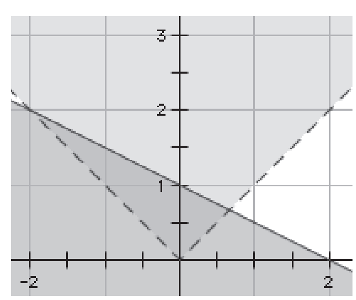
Some inequalities can be expressed with either x or y on the left side, so more than one answer may be possible for a particular graph.

Q8 Study each graph below and decide on a system of inequalities that will produce it. Check each answer on a new page of your document.

a.



b.



Q9 You probably expressed your answers to the preceding question in a form with y alone on the left side of each inequality. Where possible, rewrite your answers so that x is alone on the left side.

Q10 Use inequalities to graph an interesting shape in Sketchpad. Then challenge a friend to guess the inequalities you used.

Objective: Students graph a single inequality and combine it with a second inequality to form the graph of a system of inequalities. They then graph several more such systems.

Student Audience: Algebra 2

Prerequisites: None

Sketchpad Level: Intermediate. This activity is easier if students have already used custom tools.

Activity Time: 35–45 minutes. To shorten the time, assign only selected problems in Q6 rather than all the problems.

Setting: Paired/Individual Activity (no sketch required, but uses tools from **Inequality Tools.gsp**) or Whole-Class Presentation (use **Systems of Inequalities Present.gsp**)

Related Activity: Graphing Inequalities in Two Variables

A SINGLE INEQUALITY

The graphing tools used in this activity require inequalities (like $y > x - 2$ or $x \leq y^2 - 5$) that have been "solved" for either x or y. Most of the inequalities in this activity are already in this form. In Q7 students are asked to do the "solving" themselves. Use Q7 as a chance to review the properties of inequality needed to convert complicated expressions to the form required by the tools.

Q1 The graph crosses the y-axis at -2 and the x-axis at 1.33 and goes up and to the right with a slope of 1.5.

Q2 The inequality says that y must be less than the value of the function just graphed, so the graph of the inequality must be below the line. It extends infinitely down and to the right.

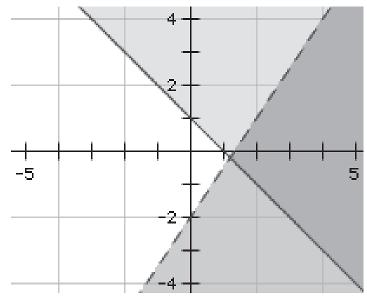
5. To avoid the need to increase the number of samples for every graph, hold down the Shift key and choose **Edit | Advanced Preferences.** On the Locus panel, change the number of point-locus samples from 300 to about 1000. (The exact value required depends on the size of the sketch window.)

Q3 The line itself is not included in the graph because y cannot be equal to the value of the function. For this reason it should appear as a dashed line.

SYSTEMS OF INEQUALITIES

Q4 Answers will vary; the important thing is that students make a conjecture before constructing the graph. (It's actually the region of the plane above and to the right of the line $x = -y + 1$.)

Q5 The graph is a portion of the plane bounded by the two lines, shown by the darkest area in the figure. This graph is infinite in extent.



Q6 The area where the two shades overlap is the solution to the system. The graphs in problem e do not overlap, so it has no solution. In problem h, have students set the angle units to radians before graphing.

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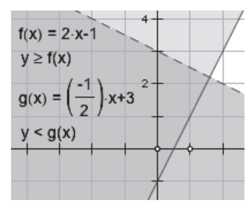

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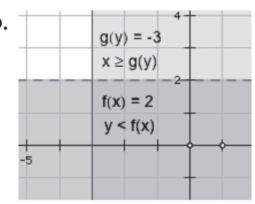

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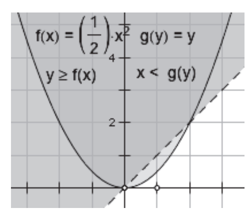

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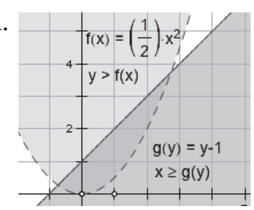

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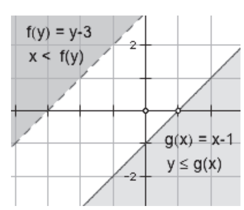

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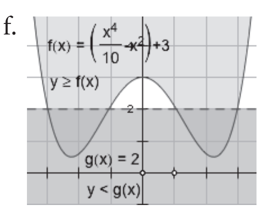

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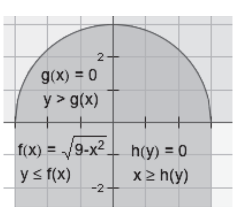

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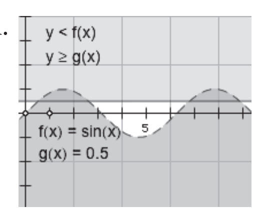
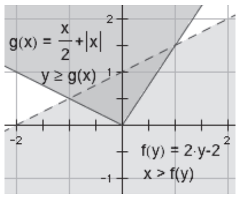
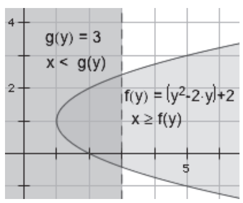


EXPLORE MORE

Q7 a. The first inequality can be expressed as $x > 2y - 2$ or as $y < 1/2x + 1$. The second inequality can be expressed as $y \geq x/2 + |x|$.



b. The first inequality can be expressed as $x \geq y^2 - 2y + 2$. The second inequality can be expressed as $x < 3$.



Q8 For some of the inequalities, it's possible to express the answers with either x or y on the left side. Here is one answer for each graph, with y on the left side of each inequality.

a. $y < 0.5x + 1$, $y > x - 1$, and $y \geq -2x - 1$

b. $y > |x|$ and $y \leq -0.5x + 1$

Q9 It's not possible to rewrite the inequality $y > |x|$ in a simple form with x on the left side. Here are the results for rewriting each of the other inequalities, with the original form on the left and the rewritten form on the right:

$y < 0.5x + 1$	$x > 2y - 2$
$y > x - 1$	$x < y + 1$
$y \geq -2x - 1$	$x \geq -0.5y - 0.5$
$y \leq -0.5x + 1$	$x \leq -2y + 2$

Q10 Answers will vary. Consider asking a few students to show and explain their answers to the class.

WHOLE-CLASS PRESENTATION

Use **Systems of Inequalities Present.gsp** to present this activity to the class. The presentation sketch first reviews graphing the single inequality $y \leq 3/2x - 2$.

1. On page 1, remind students that the first step in graphing an inequality is to graph the related equation. Use the button to show the function plot.

Q1 Ask students to predict where the graph of the inequality will be.

2. Hide the function plot and show the inequality.

Q2 Ask students why the boundary of the graph is a dashed line. Review the convention that a boundary that is part of the graph is shown as a solid line, and a boundary that is not part of the graph is shown dashed.

3. Go to page 2 to add the second inequality.

Q3 Ask students to predict where the graph will be. Have them explain how the graph of the inequality is related to the graph of $y = -x + 1$. Use Sketchpad's **Line** tool to draw an arbitrary line and have students tell you where to drag it to indicate the boundary.

Q4 Once students agree on where to place the boundary, ask them which side of the boundary the graph will be on. Use the tracer provided to check the result.

4. Press *Show* $y >= -x + 1$ to show the result.

Q5 Ask students, "Why is this boundary solid?"

Q6 Ask students, "Where are the points that satisfy both inequalities? What is the solution region?"

5. Go to page 3, which contains a different system of inequalities.

Pages 3 through 8 contain various systems of inequalities for students to practice on. Use as many of these pages as are appropriate. Be sure to have students make a prediction before showing them the result on the computer.

Q7 Ask, "What will the first inequality look like?"

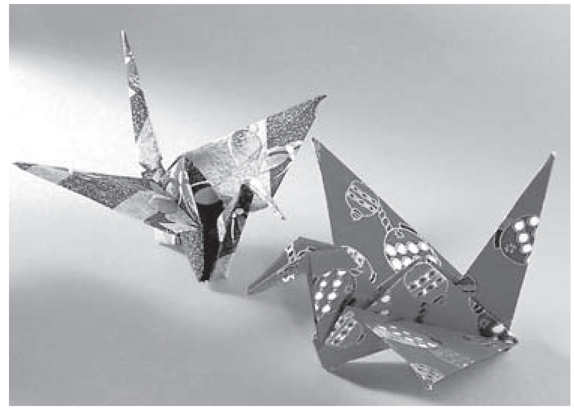
6. Use the buttons to show first the related function and then the inequality.

7. Similarly, ask about and show the second function and inequality, and then discuss the solution.

Pages 9 and 10 contain graphs of systems of inequalities. (These are the graphs from Q8 in the activity.) Use either page as a challenge to students to come up with a system of inequalities that matches the graph. Use the custom tools in **Inequality Tools.gsp** to verify student answers.

Linear Programming: Swans and Giraffes

In business, it's often difficult to figure out how to maximize profit because there are so many factors to consider: labor costs, time constraints, production capacity, etc. *Linear programming* is a branch of algebra that helps deal with complicated situations such as this.



Here's a typical linear programming problem:

Origami is the ancient Japanese art of paper folding.

Rei volunteers to bring origami swans and giraffes to sell at a charity crafts fair. It takes her three minutes to make a swan and six minutes to make a giraffe. She plans to sell the swans for \$4 each and the giraffes for \$6 each. If she has only 16 pieces of origami paper and can't spend more than one hour folding, how many of each animal should Rei make to maximize the charity's profit?

ASSIGN VARIABLES AND WRITE THE EQUATION

There's a lot of information to deal with here. The actual question is "How many of each type of animal should Rei make?" so start by assigning variables:

Let s = the number of swans Rei makes.

Let g = the number of giraffes Rei makes.

In other words, if Rei makes s swans and g giraffes, how much money will that bring in?

Q1 Write an equation for profit in terms of s and g.

1. Open **Linear Programming.gsp.** You'll see the profit equation calculated using the coordinates of point R. Each location of point R is a potential solution to the problem. For example, (6, 8) represents a solution of six swans and eight giraffes. (Does the profit equation agree with the one you came up with?)

Q2 Drag point R around and observe how the coordinate and profit measurements change. Describe how the profit relates to R's location in the plane. Where is profit greatest? Where is it smallest?

2. Choose **Graph | Snap Points** to turn on point snapping. Drag R again.

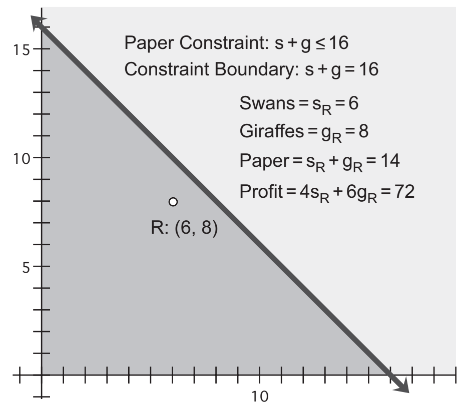
These limitations are *implicit constraints:* constraints implied in the problem but not explicitly mentioned.

Q3 What is the effect of point snapping? Why does it make sense to use it for this problem? In addition to point snapping, what other constraints (limitations) should there be when you drag R? Why do your proposed constraints make sense in terms of the original problem?

SET UP THE CONSTRAINT INEQUALITIES

The more origami figures Rei makes, the greater the profit will be. But this ignores the *constraints* (limitations) listed in the problem. The first constraint is that she has only 16 pieces of origami paper. We'll call this the *paper constraint* and express it mathematically with the *constraint inequality* $s + g \leq 16$.

3. Go to page 2. You'll see a new line and a new calculation. The line $s + g = 16$ defines the boundary of the paper constraint, and divides the first quadrant into two regions. The calculation, $s_R + g_R$ shows how many pieces of paper are used for the current point R.



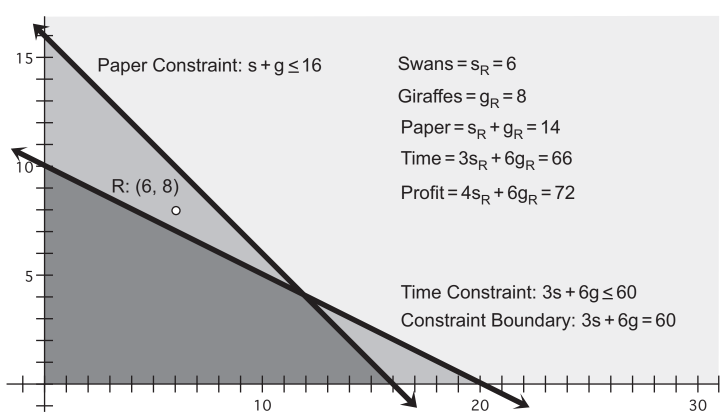
Q4 Where in the sketch can you drag R and have the paper constraint satisfied? Within that region, where is the profit greatest?

In the next two steps you'll develop a similar inequality for the *time constraint.*

Q5 The problem states: "It takes her three minutes to make a swan and six minutes to make a giraffe." Use this information to write two expressions: one for how long it takes Rei to make s swans, and the other for how long it takes her to make g giraffes.

Q6 Use your expressions from Q5 to write a constraint inequality for the time constraint—that Rei can't spend more than one hour folding.

4. Go to page 3. This page includes a blue line for the limit of the time constraint. This line and the red line divide the first quadrant into four regions, each with a different color.



Q7 For each of the four regions, list which of the two major constraints (paper and time) are met and which aren't. In which region(s) can you drag *R* and have both major constraints met?

FIND THE MAXIMUM

The region you found in Q7 in which both constraints are satisfied is called the *feasible region.* Your goal now is to find the one point within this region that maximizes profit. You can drag point *R* around and try to find where the highest profit occurs, but there's an easier way. Linear programmers can prove that *the maximum value always occurs at a corner point of the feasible region.* But which corner is it?

5. To investigate, first attach point *R* to the feasible region. To do so, select both *R* and the feasible region, and choose **Edit | Merge Point To Quadrilateral.**

Q8 Drag point *R* around the perimeter of the feasible region, noting the profit value at the corner points. How many swans and giraffes should Rei make in order to maximize the profit? What is the maximum profit?

EXPLORE MORE

Q9 Suppose that Rei actually has 18 pieces of paper. Change page 3 of the sketch to investigate this new problem. (To do so, press the *Show Parameters* button, then double-click *Sheets of Paper* and change its value.) How does this change the feasible region? Can Rei increase profits with this new feasible region?

Q10 Continue experimenting with changing the paper constraint. How much does the maximum profit increase with each extra sheet?

Q11 Rei was limited by only two major constraints. What if a third were added? For instance, what if Rei didn't want to make more than eight swans? Turn this new constraint into a constraint inequality and describe the new feasible region as well as the maximum profit.

Q12 Go to page 4 of **Linear Programming.gsp.** You'll see a new line labeled *Profit.* Alternate between dragging point *R* along this new line and dragging the line itself. (Drag the line by dragging its *y*-intercept.) Explain what the new line represents. Then explain how it demonstrates why profit will always be maximized at a corner point of the feasible region.

Objective: Students explore a linear programming problem. They turn the problem statements into constraint equations, explore the feasible region, and determine how to maximize a desired quantity.

Student Audience: Algebra 2

Prerequisites: Students should be fluent with linear equations, simultaneous equations, and word problems.

Sketchpad Level: Easy

Activity Time: 40–50 minutes

Setting: Paired/Individual Activity (use **Linear Programming.gsp**) or Whole-Class Presentation (use **Linear Programming Present.gsp**)

Related Activity: Graphing Systems of Inequalities

ASSIGN VARIABLES AND WRITE THE EQUATION

Q1 Profit $= 4s + 6g$

Q2 The closer point R is to the origin, the less the profit. The farther up and to the right, the greater the profit. (Some students may notice already that the relationship is a bit more subtle than this, because the profit at (0, 16) is more than the profit at (16, 0). This observation becomes important when answering Q4.)

Q3 It makes sense to have point R snap to locations with integer coordinates because Rei can sell only whole numbers of swans and giraffes. It doesn't make sense to consider her making 3.71 swans or 4.2 giraffes.

Besides limiting both s and g to integers, there are two more implicit restrictions: $s \geq 0$ and $g \geq 0$: Rei cannot make negative numbers of either animal. These restrictions limit R to the first quadrant.

SET UP THE CONSTRAINT INEQUALITIES

Q4 The paper constraint is satisfied only within (and on the perimeter of) the triangle bounded by the two axes and the purple line. Within this region, the profit is greatest at the upper-left corner, or (0, 16). This makes sense since, without the time constraint, the greatest profit would be made by using all 16 pieces of paper to make giraffes.

Q5 $3s =$ how long it takes Rei to make s swans
$6g =$ how long it takes Rei to make g giraffes

Q6 $3s + 6g \leq 60$

Q7 In the yellow region, neither constraint is met. In the light green region, only the time constraint is met. In the red region, only the paper constraint is met. In the dark blue region, both are met.

FIND THE MAXIMUM

Q8 Rei should make 12 giraffes and 4 swans for a profit of \$72.

EXPLORE MORE

Q9 This increases the size of the feasible region a bit, moving the upper-right vertex a little down and to the right. Rei would be able to make a greater profit with this change. In fact, increasing the number of pieces of paper Rei has would increase her profit up to the point at which she has 20 pieces. Having more than 20 pieces of paper wouldn't make any difference (because of the time constraint).

Q10 Each additional piece of paper increases her profit by \$2, to a maximum of \$80 profit with 20 pieces of paper.

Q11 The line $g \leq 8$ corresponds to the constraint "Rei didn't want to make more than eight swans." This is a region to the left of a vertical line that cuts the original feasible region into a smaller pentagonal region. The maximum profit for this situation is \$68, occurring again at the upper-right vertex of the feasible region: eight swans and six giraffes.

Q12 The profit line represents all combinations of giraffes and swans that give the same profit. To see this, drag point R along the line: The profit doesn't change. With the profit line far up and to the right, you get a large profit, but R doesn't touch the feasible region. With the profit line cutting through the feasible region, R can take on feasible values, but the profit isn't at its greatest. Only when the profit line just touches a corner of the feasible region is the profit maximized.