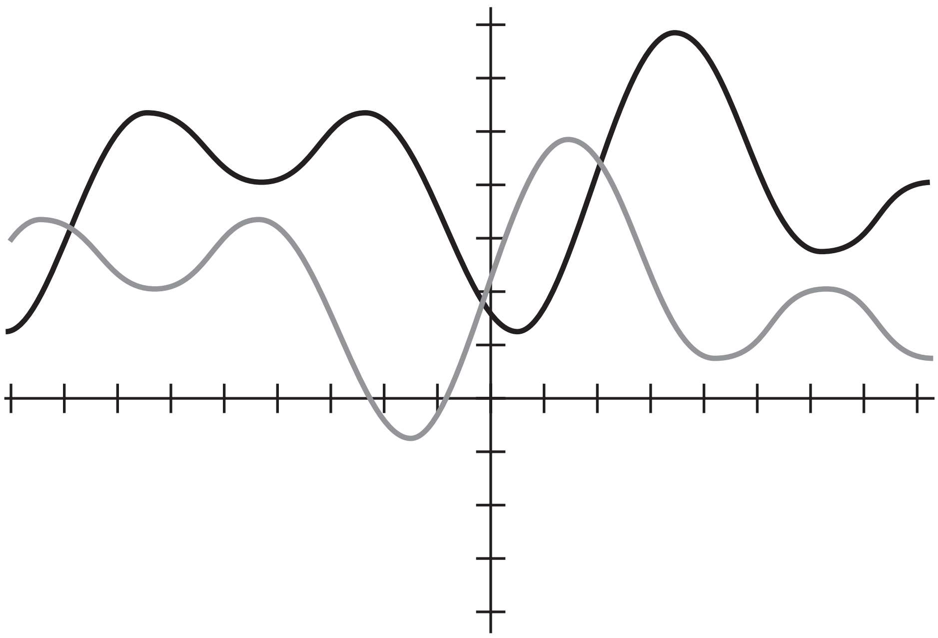
5

Algebraic Transformations



Translating Coordinates

Analytic geometry is a branch of mathematics that uses a coordinate system to study geometry. In this activity you'll do some analytic geometry as you study what happens to the coordinates of points—specifically, the vertices of a triangle—when they're translated.

SKETCH

Choose **Graph | Define Coordinate System.** Then choose **Graph | Snap Points.**

You'll start by creating a coordinate system and drawing the triangle to be translated.

1. In a new sketch, create a new coordinate system with point snapping turned on.
2. All the coordinate measurements in this activity will be integers, so change Sketchpad's preferences to display the coordinates to the nearest integer. Choose **Edit | Preferences,** and on the Units panel set the Precision for Scalars (Slope, Ratio, . . .) to **units.**
3. Using the **Segment** tool, draw a triangle whose vertices are grid points.

Select the three points with the **Arrow** tool, and choose **Measure | Coordinates.**

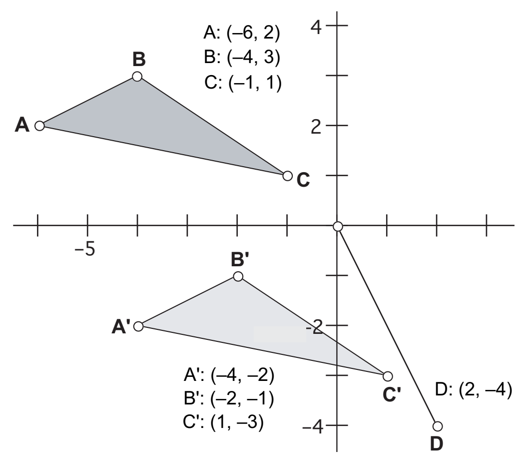
4. Measure the coordinates of the triangle's three vertices. The vertices are labeled *A, B,* and *C.*
5. Construct the triangle's interior by selecting its three vertices and choosing **Construct | Triangle Interior.**

Now you need to tell Sketchpad how to translate the triangle. To do this, you'll construct a segment and use it to define a *translation vector.*

6. Draw a segment from the origin to a point *D* in the plane.
7. Measure the coordinates of point *D.* Drag this new coordinate measurement so that it's away from the others.

Select in order the origin and point *D.* Choose **Transform | Mark Vector.**

8. Mark the vector from the origin to point *D.* A brief animation shows the vector that you've marked.



Now translate the triangle and measure the new coordinates.

9. Translate the triangle by the marked vector by selecting the entire triangle (vertices, sides, and interior) and choosing **Transform | Translate.**

10. Measure the coordinates of the new triangle's vertices.
11. Experiment by dragging point D or any of the triangle vertices. Look for a relationship between a point's coordinates, the coordinates of its image under a translation, and the coordinates of point D.

INVESTIGATE

Q1 Where can you drag point D so that the original points and their corresponding image points always have the same y-coordinates? The same x-coordinates?

Q2 When the vector you defined translates the triangle to the left and up, in what quadrant is point D? How do you know?

Q3 Suppose point D has coordinates (s, t). What are the coordinates of the image of a point (x, y) under a translation by (s, t)?

Q4 Quadrilateral $JKLM$ has vertices J: $(-2, -1)$, K: $(-3, 3)$, L: $(-1, 1)$, and M: $(1, 2)$. In its translated image $J'K'L'M'$, J' is at $(1, -2)$, K' is at $(0, 2)$, and L' is at $(2, 0)$. Where is M'?

Q5 In the previous problem, assume that $JKLM$ was translated to $J'K'L'M'$ using the vector in the sketch. What are the coordinates of point D?

EXPLORE MORE

Q6 Using Sketchpad's Measure menu, you can measure angles, point coordinates, segment slope and length, and perimeter and area of the interior. Which of these measurements are preserved (don't change) under a translation? Which are not preserved?

Q7 Triangle UVW has vertices U: $(5, 8)$, V: $(2, -5)$, and W: $(-1, 3)$. Triangle XYZ has vertices X: $(3, 6)$, Y: $(0, -7)$, and Z: $(-2, 0)$. Explain why XYZ cannot be the image of UVW under a translation.

Translating Coordinates

Objective: Students translate points in the coordinate plane and learn the associations between the translation and the image coordinates.

Student Audience: Algebra 1/Geometry/Algebra 2

Prerequisites: Students should understand the (x, y) notation of a point in the coordinate plane. Some previous familiarity with translations is also helpful. The term *translation vector* is used and defined. The term *image* (as in "the coordinates of its image under a translation") is used but not defined, so you may want to discuss this terminology from transformational geometry.

Sketchpad Level: Intermediate. Students start with a blank sketch and construct everything from scratch.

Activity Time: 35–45 minutes. If you have less time or are working with inexperienced Sketchpad users, you can use the sketch **Translate.gsp.** Have students open this sketch, go to the page labeled "After Step 10," and start the activity at step 11.

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Coordinate Translation Present.gsp**)

You may wish to begin this activity by discussing vectors. What are vectors? How are they different from lines, rays, or segments?

If students finish early, you might suggest that they try translating several times with the same vector or translating using two or more different vectors. How are the final images related to the translation vectors in each case? Students may also try using vectors whose tails aren't attached to the origin.

You might also ask students to extend the activity by exploring the various ways of specifying a translation in Sketchpad. This activity uses a marked vector defined by two points (its tail and head). One alternative is to specify a distance and an angle (a *polar vector*); another is to specify a horizontal distance and a vertical distance (a *rectangular vector*). Any of these three methods of specifying a vector can be used in Sketchpad, though the focus of this discussion should not be on Sketchpad functionality, but rather on various ways to specify a vector. You could ask students to discuss when they might want to use one method and when they might want to use another.

INVESTIGATE

Q1 If you drag point D along the x-axis, the y-coordinates of a vertex and its image point will be equal. If you drag point D along the y-axis, the x-coordinates of a vertex and its image point will be equal.

Q2 If the vector translates the triangle to the left and up, point D is in the second quadrant. Its x-coordinate is negative (causing the leftward movement) and its y-coordinate is positive (causing the upward movement).

Q3 The image of (x, y) under a translation by (s, t) is the point $(x + s, y + t)$. Note that this answer holds regardless of the signs of x, y, s, or t.

If this question is difficult for students, have them use actual values for x, y, s, and t, all positive at first, and have them look for patterns.

Q4 Point M' will be at $(4, 1)$. The reason is that the translation vector here is $(3, -1)$, three to the right and one down. $(1 + 3, 2 + (-1))$ is $(4, 1)$.

Q5 Point D is at $(3, -1)$.

EXPLORE MORE

Q6 All of the measurements are preserved except for the point coordinates.

Q7 Triangle XYZ cannot be the image of triangle UVW under translation because the three translation vectors aren't the same. The vectors used to get from U to X and from V to Y are both $(-2, -2)$, whereas the vector used to get from W to Z is $(-1, -3)$. Other arguments are possible (for example, the two triangles aren't similar).

This presentation uses coordinates to represent three major components: a triangle in the coordinate plane, a translation vector, and the translated image of the triangle. In Q4–Q6, you will remove one of these things from view and challenge the class to derive it from the two remaining. Do plenty of repetitions, and be sure to give students a look at some negative numbers.

1. Open **Coordinate Translation Present.gsp.** Press *Translate*. A quick animation will show the translation of the triangle by the vector from the origin to point *D*.

Spend a few minutes discussing the meaning of translation. As you speak, drag points *A*, *B*, *C*, and *D*. The image will move accordingly. Students will tend to pick up on the geometric relationships better when they see the objects in motion. This would be a good opportunity to acquaint them with the terms *pre-image* ($\triangle ABC$), *image* ($\triangle A'B'C'$), and *translation vector* ($\overrightarrow{OD}$).

Q1 What is the image of point *A*? (A')

Q2 Concentrate on the coordinates of points *A*, A', and *D*. What relationship is there between these numbers? (Give plenty of time for a response. $x_A + x_D = x_{A'}$ and $y_A + y_D = y_{A'}$)

Q3 What about points *B* and *C* and their images? What's the connection? (They have the same association. The differences between corresponding coordinates of a point and its image are defined by the coordinates of point *D*.)

2. Press *Secret Translation*. This hides all of the objects and moves the pre-image and point *D* to new locations. By construction, the image must move also.

3. Show the pre-image and point *D*.

Q4 What are the coordinates of the image points? (Give students time to work it out, and then press *Show Image*.)

4. Press *Secret Translation* again. This time show the pre-image and the image.

Q5 What are the new coordinates of point *D*? (Press *Show Point D*.)

5. Press *Secret Translations* again. Show *D* and the image.

Q6 What are the coordinates of the pre-image points? (Press *Show Pre-image*.)

When you think students are ready, turn off the presentation and do the same thing on a board, this time using coordinates only—no sketches.

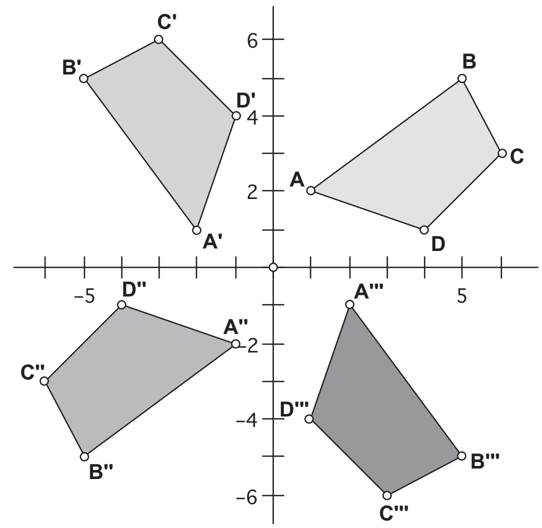
Rotating Coordinates

In this activity you'll investigate what happens to the coordinates of points after you rotate them about the origin, mainly by multiples of 90°. If you do the Explore More exercises, you'll do the same thing in polar coordinates.

SKETCH AND INVESTIGATE

Choose **Graph | Define Coordinate System** and then **Graph | Snap Points.**

1. In a new sketch, create a new coordinate system with point snapping turned on.
2. Using the **Segment** tool, draw a quadrilateral whose vertices are grid points. Use the **Text** tool to label the vertices *A, B, C,* and *D.*



3. Select the four vertices consecutively clockwise or counter-clockwise, and choose **Construct | Quadrilateral Interior.**
4. Mark the origin of your coordinate system as a center of rotation by selecting it and choosing **Transform | Mark Center.**

To rotate one or more objects, select them and choose **Transform | Rotate.** Enter the angle to use in the dialog box that appears. Click Rotate when the settings are correct.

5. Rotate the entire quadrilateral (points, sides, and interior) by 90°.
6. Drag a vertex of the original quadrilateral, and observe how the image responds.
7. Now rotate the image quadrilateral by 90°. Rotate the next image too. You should have four quadrilaterals on the screen.

Q1 If you were to rotate the fourth quadrilateral, where would its image be?

8. Select point *A* and its three rotated images (its *corresponding* points on the other three quadrilaterals). Choose **Measure | Coordinates.**

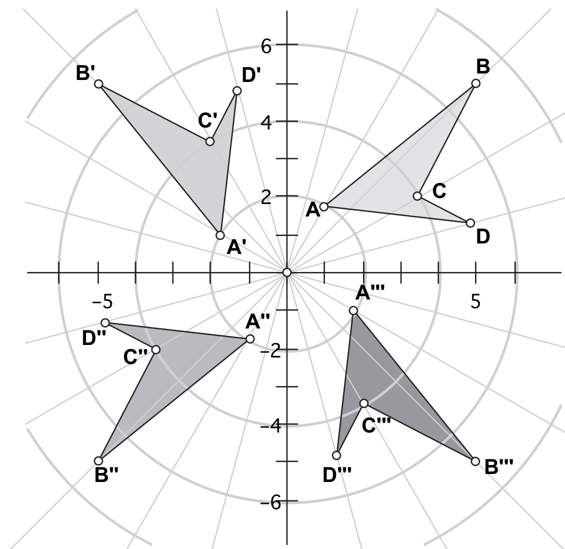
Q2 What is the relationship between the coordinates of a point and the coordinates of its image point after a 90° rotation about the origin? Suppose that a point with coordinates (*a, b*) is rotated by 90° about the origin. What are the coordinates of its image point?

Q3 What is the relationship between the coordinates of a point and the coordinates of its image point after a 180° rotation about the origin? If a point with coordinates (*a, b*) is rotated by 180° about the origin, what will the coordinates of its image point be?

Q4 If you rotate an object by $-90°$, it will be rotated clockwise rather than counter-clockwise. Suppose you rotate a point with coordinates (a, b) by $-90°$. What will the coordinates of the rotated point be?

Q5 What are the coordinates of *B*? Without doing any other measurements or even looking at the screen, what are the coordinates of the three rotated images of *B*?

EXPLORE MORE



Q6 Using Sketchpad's Measure menu, you can measure angles, point coordinates, a segment's slope and length, and a polygon's perimeter and area. Which of these measurements are preserved (don't change) under a rotation? Which are not preserved?

9. Many people prefer to use a grid for certain kinds of graphing. To get a polar grid, choose **Graph | Grid Form | Polar.** Now once again measure the coordinates of a point and its image points. Then drag the points around to determine the relationship between the coordinates of rotated points.

Q7 Again suppose that a point has coordinates (a, b), but in this case they are polar coordinates. What are the coordinates of the images when you rotate this point 90° about the origin three times in succession?

Q8 Rotate a point by 45° about the origin. Measure the coordinates of the original point and its image point in both a square coordinate system and a polar coordinate system. In which type of coordinate system is the relationship between the coordinates clearer?

Objective: Students learn some basics of coordinate rotation by rotating figures about the origin by multiples of 90°.

Student Audience: Algebra 1/Geometry/Algebra 2

Prerequisites: Students should understand the (x, y) notation of a point in the coordinate plane. Some previous familiarity with rotation is also helpful. The term *image* (as in "the coordinates of its image point after a 90° rotation") is used but not defined, so you may want to discuss this terminology from transformational geometry.

Sketchpad Level: Intermediate. Students start with a blank sketch and construct everything from scratch. All steps are clearly explained (and extra tips are given below), but if students are unfamiliar with Sketchpad, the activity might take longer than the estimate given.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Coordinate Rotation Present.gsp**)

While this activity is progressing, you may want to ask students why only multiples of 90° are being considered. The Explore More section addresses this issue, but students may not get that far and the issue is worth thinking about before then anyway.

A good extension to this activity involves asking students why the relationships in this activity hold. Consider suggesting that students form the triangles defined by the origin, a point, and the point's image.

As another extension, you could discuss whether these relationships will hold if the center of rotation is a point other than the origin.

CONSTRUCTION TIPS

The instructions call for the Snap Points setting. It's not actually necessary to change this, but the coordinate relationships will be easier for students to see if they work with integers.

SKETCH AND INVESTIGATE

Q1 A fourth rotation would map the image to the original pre-image quadrilateral.

Q2 If you switch the x- and y-coordinates of the original point, then change the sign of the new x-coordinate, you get the coordinates of the image point after a 90° rotation about the origin. The coordinates of the image point are $(-b, a)$.

Q3 Both the x- and y-coordinates of the image point are the opposite of those of the original point. The coordinates of the image of point (a, b) are $(-a, -b)$.

Q4 The coordinates of the image point are $(b, -a)$.

Q5 The answer will depend on the coordinates of point B. The coordinates of the four points will have the form $B(a, b)$, $B'(-b, a)$, $B''(-a, -b)$, and $B'''(b, -\text{a})$,

EXPLORE MORE

Q6 Angles, length, perimeter, and area are preserved. Coordinates and slope are not.

Q7 In polar coordinates, the first coordinate represents the distance of a point from the origin. That distance does not change when the origin is the center of rotation. The second coordinate, the angle from the positive x-axis, changes in increments of 90°. Starting with point $A(a, b)$, the images should be in the form $A'(a, a + 90°)$, $A''(a, a + 180°)$, and $A'''(a, a + 270°)$.

Q8 In a square coordinate system, it's difficult to see the relationship between the coordinates of a point and those of its image under a 45° rotation. In a polar coordinate system, the relationship is simple: The r-coordinates are the same, and the θ-coordinate of the image point is that of the original point plus 45°.

To begin this presentation, explain that you intend to investigate some relatively simple rotations. The origin will be the center of rotation, and all rotations will be multiples of 90°.

1. Open **Coordinate Rotation Present.gsp.**
2. Press the Rotate button. This generates rotations of the quadrilateral by angles of 90°, 180°, and 270°. Drag some of the vertices on the red quadrilateral to show how the rotations respond.
3. Take a moment to introduce or review the transformation notation used here. When a point is transformed, a prime ($'$) symbol is added to the label of its image, which is why all of the point labels on the last image have three prime symbols.
4. Press *Show A*.

Q1 These are the coordinates of point A. What should the coordinates be for A'? The actual answer will depend on the coordinates of A. Given A: (a, b), the coordinates of A' will be $(-b, a)$.

Go through each of the Show buttons one at a time to see all three images of A. Pause each time to ask for predictions. The general form will be as follows:

$$A(a, b), \quad A'(-b, a), \quad A''(-a, -b), \quad A'''(b, -a)$$

5. Press *Show B*.

Q2 What are the coordinates for the three images of point B? Give students time to agree on an answer for all three before revealing the coordinates. They should anticipate the same pattern.

Page 2 is actually the same sketch, but with the settings changed. Walk through the same presentation again, and ask the class to predict coordinates on the rotated points. This progression follows a different, simpler pattern:

$$A(a, b), \quad A'(a, b + 90°), \quad A''(a, b + 180°), \quad A'''(a, b + 270°)$$

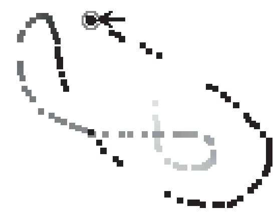
All of the θ-coordinates will be reduced to equivalent angles between $-180°$ and 180°.

If time allows, press *Reset,* hide the coordinates, and drag vertices to start again with a different quadrilateral. Do this on both pages. Try putting the pre-image somewhere other than the first quadrant.

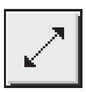
Reflecting in Geometry and Algebra

If you're like most people, you've spent at least a little time looking at yourself in the mirror, so you're already familiar with reflection. In this activity you'll add to your knowledge on the subject as you explore reflection from both geometric and algebraic perspectives.

SKETCH AND INVESTIGATE

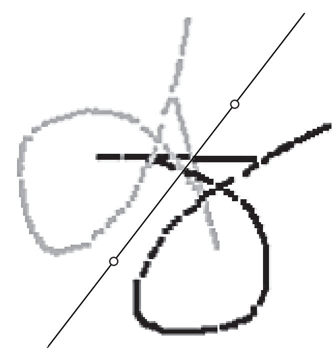


1. In a new sketch, use the **Point** tool to draw a point.
2. With the point still selected, choose a color from the **Display | Color** submenu. Then choose **Display | Trace Point.** Use the **Arrow** tool to drag the point around. The trail the point leaves is called its *trace.*
3. If the trace remains on the screen without fading, choose **Edit | Preferences.** On the Color panel, check Fade Traces Over Time and click OK.



To choose the **Line** tool, press and hold the mouse button on the **Straightedge** tool, then drag and release over the **Line** tool in the palette that appears.

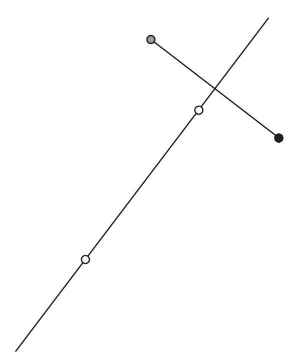
4. Using the **Line** tool, draw a line. With the line selected, choose **Transform | Mark Mirror.** An animation indicates that the mirror line has been marked.
5. Using the **Arrow** tool, select the point. Choose **Reflect** from the Transform menu. The point's reflected image appears.
6. Give the new point a different color, and turn on tracing for it as well.



Starting in this step, we'll refer to the two points defining the line as *line points* and the other two points as *reflecting points.*

7. What will happen when you drag one of the reflecting points? Ponder this a moment. Then drag and see. What do you think will happen when you drag one of the line points? Find the answer to this question too.

Q1 Briefly describe the two types of patterns you observed in step 7 (one when dragging a reflecting point, the other when dragging a line point).



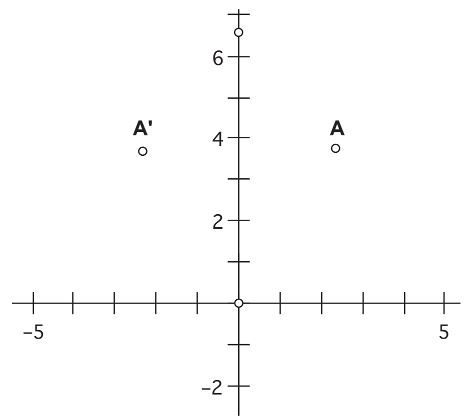
8. Select the reflecting points; then choose **Display | Trace Points** to toggle off tracing.
9. With the two points still selected, choose **Construct | Segment.** A segment is constructed between the points. Drag the various objects around and observe the relationship between the line and the segment.

Q2 What angle do the line and the segment appear to make with each other? How does the line appear to divide the segment?

FROM GEOMETRY TO ALGEBRA

Now that you've reviewed some geometric properties of reflection, you can apply this knowledge to reflection in the *xy* plane. Start by exploring reflection across the *y*-axis.

10. Click in blank space to deselect all objects. Drag one of the line points so it's near the center of the sketch. With this point selected, choose **Define Origin** from the Graph menu. A coordinate system appears. The selected point is the origin, (0, 0).



11. Deselect all objects; then select the *y*-axis and the other line point (the one that didn't become the origin). Choose **Edit | Merge Point To Axis.** The point now attaches itself to the *y*-axis, which acts as the mirror line.

12. Drag one of the reflecting points and consider the coordinates of both points as they move. How do the coordinates of the two points compare?

13. Select both of the reflecting points. Choose **Measure | Coordinates.**

Q3 A point with coordinates (*a*, *b*) is reflected across the *y*-axis. What are the coordinates of its reflected image?

14. Now consider the distance between the two reflecting points and how it relates to their coordinates. Make a prediction. Then select the two points and choose **Measure | Coordinate Distance.**

A special challenge is to make sure your answers to this question and Q6 work regardless of what quadrants the points are in.

Q4 A point with coordinates (*a*, *b*) is reflected across the *y*-axis. How far is it from its reflected image?

15. Deselect all objects. Then select the point on the *y*-axis that you merged in step 11. Choose **Split Point From Axis**.

The point is split from the *y*-axis.

16. With the point still selected, select the *x*-axis as well. Then choose **Edit | Merge Point To Axis** from the Edit menu. The *x*-axis now acts as the mirror line. Drag one of the reflecting points and observe the various measurements.

Q5 A point with coordinates (a, b) is reflected across the x-axis. What are the coordinates of its reflected image?

Q6 A point with coordinates (a, b) is reflected across the x-axis. How far is it from its reflected image?

EXPLORE MORE

17. Plot the line $y = x$. Split the point from the x-axis and merge it to the new line.

Q7 A point with coordinates (a, b) is reflected across the line $y = x$. What are the coordinates of its reflected image?

18. Consider the following transformations (each is separate):
    a. Reflect a point over the x-axis, and then reflect the image over the y-axis.
    b. Reflect a point over the y-axis, and then reflect the image over the x-axis.
    c. Rotate a point by 180° about the origin.

Q8 How do these three transformations compare? What would the coordinates of a point (a, b) be after each of these transformations?

Objective: Students reflect points across the coordinate axes and learn the algebraic associations between the coordinates of the point and its image.

Student Audience: Algebra 1/Geometry/Algebra 2

Prerequisites: Familiarity with the Cartesian plane

Sketchpad Level: Easy/Intermediate. The constructions steps that students do are explained in detail.

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Reflection Present.gsp**)

Related Activities: Reflecting Functions

This activity works well as a brush-up for students having problems with the coordinate plane, as an introduction to using Sketchpad for both geometry and algebra, and as preparation for function transformation in later activities.

Before starting this activity, it would be a good idea to brainstorm what students already know about reflection. A useful question to discuss—anticipating Q4 and Q6—is "If you stand 3 feet from a mirror, how far do you *appear* to be from your reflected image?"

During the activity you may wish to encourage students to continually connect the geometric and algebraic realms. For example, how do the relationships found in Q4 and Q6 relate to those from Q2?

If your class has done this and the previous two activities, you may want to end this activity by comparing all three transformations—translation, rotation, and reflection. Which transformation was the easiest to understand from an algebraic perspective? Which was the most surprising?

CONSTRUCTION TIPS

3. You may choose instead to leave Fade Traces Over Time unchecked so that traces remain on screen to be examined. In this case, students would periodically need to choose **Erase Traces** from the Display menu to clear traces from their screens.

14. The reason for using **Coordinate Distance** instead of **Length** or **Distance** from the Measure menu is that the scale of the axes can be changed. A unit on the coordinate grid does not necessarily have the same length as the distance measurement unit.

SKETCH AND INVESTIGATE

Q1 Dragging a reflecting point results in a mirror pattern with the two traces mirroring each other across the line. Dragging a line point causes the reflected image point to draw a circle around the other line point. The radius of this circle is the distance between the other line point and the reflecting pre-image.

Q2 The line is the segment's perpendicular bisector, meaning that the angle they form is 90° and the line cuts the segment in half.

FROM GEOMETRY TO ALGEBRA

Q3 $(-a, b)$

Q4 The distance is $|2a|$ or $2|a|$, but give some credit for the answer $2a$. The absolute value signs ensure that the answer will be positive even if a is negative. This is desirable because the distance between two objects is always considered to be non-negative.

Q5 $(a, -b)$

Q6 $|2b|$ or $2|b|$, and maybe $2b$ (See the answer to Q4 above.)

EXPLORE MORE

Q7 The coordinates switch places. The image of a point (a, b) reflected across the line $y = x$ is the point (b, a).

Q8 These three transformations are equivalent. The coordinates of (a, b) after any of the transformations are $(-a, -b)$.

1. Open **Reflection Present.gsp.** Drag point P as you explain that P' is its reflection across the blue line. Also drag the points that define the line. Experiment with the *Show Traces* button to emphasize the reflection properties.

2. Construct line segment PP'.

Q1 What is the relationship between this line segment and the mirror line? (The mirror line is the perpendicular bisector of the line segment.)

3. Open page 2. This is essentially the same construction moved to the coordinate plane. Now you can control the mirror line only by using the action buttons. Press *Reflect Over x-axis* to align the mirror line with the x-axis.

All of the answers on this page are with respect to coordinates (a, b) for point P.

Q2 The coordinates of point P are on the screen. Ask, "Knowing what you do about the properties of reflection, what are the coordinates of P'?" (For $P(a, b)$, the coordinates of P' are $(a, -b)$. Press *Show Coordinates of P′*.)

Q3 What is the distance PP'? ($|2b|$)

4. Press *Hide Coordinates of P′*, *Hide PP′*, and *Reflect Over y-axis.*

Q4 Now what are the coordinates of P'? $(-a, b)$

Q5 What is the distance PP'? $|2a|$

5. Hide the measurements again, and press *Reflect Over* $y = x$. This one is more difficult, so take it slowly.

Q6 What are the coordinates of P'? (b, a)

Q7 What is the distance PP'? $|\sqrt{2}(a - b)|$

6. Press *Reflect Over x-axis.*

Q8 Here it is reflected on the x-axis again. Ask, "What will you get if you reflect this image, P', across the y-axis?" (The second image is $P''(-a, -b)$, and $PP'' = 2\sqrt{a^2 + b^2}$.)

7. After plenty of discussion, press *Show Second Reflection, Show Coordinates of P″*, and *Show PP″*.

Q9 Ask, "What happens if you reflect it on the y-axis first and then the x-axis?"

8. Leaving all of the objects still showing, press *Reflect Over y-axis.* Both of the mirror lines will rotate together, and as this happens, students will see that P'' does not move at all. This should lead to a general conclusion about reflections on two perpendicular lines.

Stretching and Shrinking Coordinates

In geometry dilation is defined in terms of moving toward or away from a center point. On the coordinate plane we can define an additional transformation in which an object is moved toward or away from a line, usually one of the coordinate axes. Such transformations are usually called *shrinks* and *stretches.* In this activity you'll create shrinks and stretches and investigate their properties.

STRETCH AND SHRINK

First you'll use a stretch to move a point away from the x-axis.

1. Open **Stretch Shrink Coords.gsp.** The sketch contains a coordinate system, a polygon, and a slider labeled a.

Use the **Text** tool to label the point.

To measure the x-coordinate, select the point and choose **Measure | Abscissa (x).** Use a similar method for y.

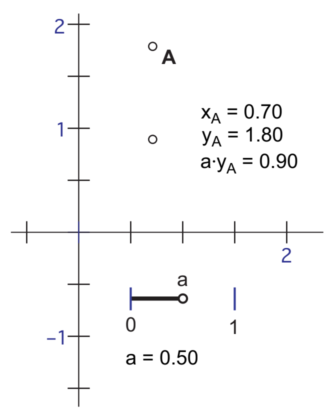
2. Construct a point using the **Point** tool. Label it A and measure its x- and y-coordinates.

Q1 Which of these two measurements tells you how far the point is from the x-axis?

Q2 How could you change the point's coordinates to define a new point that's twice as far away from the x-axis?

Enter the value y_A into the calculation by clicking on the y_A measurement in the sketch.

3. Use Sketchpad's Calculator to multiply the measured y-coordinate by 2. To do this, choose **Measure | Calculate** and calculate $2y_A$.
4. Construct the new point by selecting in order the values x_A and $2y_A$ and then choosing **Graph | Plot As (x, y).** This new point is called the *image,* and the original point A is called the *pre-image.*
5. Drag pre-image A and observe the behavior of the image.



Q3 How does the image behave when the pre-image is near the x-axis? How does it behave when the pre-image is below the x-axis?

To edit the calculation, double-click it. Then change the expression to put a (the value of the slider) in place of the 2.

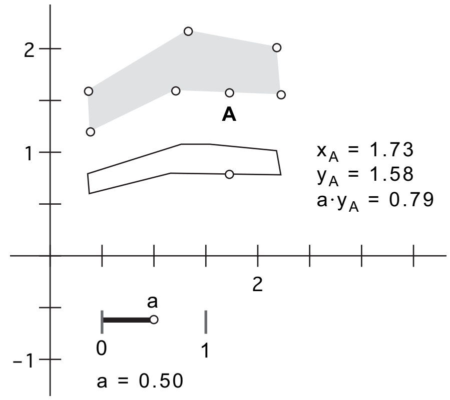
6. Edit the calculation $2y_A$ to make it $a \cdot y_A$, so that it multiplies by the value of the slider instead of by the constant 2.
7. Drag the slider to set its value to 0.50.

Q4 Drag the pre-image point. How does the image point behave now? Is this a stretch or a shrink?

Now you'll attach the point to the polygon border and use it to apply the same transformation to every point of the polygon.

8. Merge point A to the hexagon by selecting both the point and the hexagon interior and choosing **Edit | Merge Point To Hexagon.**

Q5 Drag point A around the hexagon. How does the image point behave now?



9. To construct the locus of the image as pre-image A moves around the hexagon, select both points and choose **Construct | Locus.**

Q6 Drag the slider so $a = 2$. How does this change the locus? What does the locus look like if you make $a = 1$?

10. Page 2 contains a polygon and two sliders, a and b. Construct a new point A to be the pre-image. Measure its x- and y-coordinates.
11. Use Sketchpad's Calculator to compute the new coordinates: $b \cdot x_A$ and $a \cdot y_A$.
12. Construct the image point by selecting the two calculations in order and choosing **Graph | Plot As (x, y).**
13. As you did before, attach point A to the polygon, and then construct the locus of the image point as pre-image A moves around the polygon.
14. Experiment by dragging the sliders and observing the effect on the locus.

Q7 Set the sliders so $a = 2$ and $b = 2$. Describe the shape of the locus compared to the original polygon.

Q8 Describe the shape of the locus, relative to the original polygon, for each combination of slider values below:

a. $a = 0.50$ and $b = 0.50$

b. $a = 1$ and $b = 1$

c. What conclusion can you draw when the two sliders have equal values?

Q9 Now describe the results for these slider values:

a. $a = 1$ and $b = 3$

b. $a = 0.50$ and $b = 1$

c. $a = 2$ and $b = 3$

EXPLORE MORE

Q10 Describe the transformations that result from these slider values:

a. $a = 1$ and $b = -1$

b. $a = -1$ and $b = 1$

c. $a = -0.5$ and $b = -0.5$

What can you conclude about stretches and shrinks by negative values?

Stretching and Shrinking Coordinates

Activity Notes

Objective: Students multiply the coordinates of a point to stretch or shrink a polygon toward or away from the axes. They multiply by different numbers and observe the resulting transformations. This activity lays the groundwork for stretching and shrinking functions.

Student Audience: Algebra 2

Prerequisites: It is helpful if students are familiar with geometric transformations, and particularly with dilation.

Sketchpad Level: Intermediate.

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (use **Stretch Shrink Coords.gsp**) or Whole-Class Presentation (use **Stretch Shrink Coords Present.gsp**)

Related Activities: Stretching and Shrinking Functions

STRETCH AND SHRINK

The informal definition of stretches and shrinks in the first paragraph of this activity describes them as moving points toward or away from the axes, suggesting that only a single object is involved. The activity itself uses the terms pre-image and image, implying two separate objects. This apparent contradiction—whether a transformation involves changing a single object or creating a second object related to the first—is confusing and merits class discussion. In fact, we often use both ways of talking about transformations, because a transformation can be viewed in a static way (with a pre-image giving rise to a separate image) or in a dynamic way (with a single object changing from one form to another). Each way of describing transformations is useful; encourage students to become comfortable with both.

Q1 The y-value tells how far the point is from the x-axis. (The x-value tells you how far it is from the y-axis.)

Q2 A point twice as far from the x-axis must have a y-value twice as great, so you should multiply the y-value by 2.

Q3 The image point always stays twice as far from the x-axis, no matter where you move the pre-image.

Q4 The image is now only half as far from the x-axis as the pre-image. This is a shrink.

Q5 The image point traces out a path related to the border of the polygon, but with a vertical shrink. This path has the same width as the polygon, but is only half as high.

Q6 When $a = 2$, the image point traces out a path related to the border of the polygon, but with a vertical stretch. This path is as wide as the polygon, but is twice as high. When $a = 1$, the image coincides with the pre-image.

Q7 When both a and b are 2, the image is twice as large as the original polygon, both vertically and horizontally. This is equivalent to a geometric dilation by a factor of 2.

Q8
a. When $a = 0.50$ and $b = 0.50$, the image is half the size of the original, both vertically and horizontally.

b. When $a = 1$ and $b = 1$, the image coincides with the original polygon.

c. When the two sliders have equal values, the image is transformed the same way vertically and horizontally, so the image has the same shape as the original, although it may be larger or smaller.

Q9
a. When $a = 1$ and $b = 3$, the image is stretched by a factor of 3 horizontally, but not at all vertically, so it's three times as wide as the original and the same height as the original.

b. When $a = 0.50$ and $b = 1$, the image is half as high as the original and the same width.

c. When $a = 2$ and $b = 3$, the image is twice as high as the original and three times as wide.

EXPLORE MORE

Q10
a. When $a = 1$ and $b = -1$, the image is the same size as the original, but is reflected across the y-axis.

b. When $a = -1$ and $b = 1$, the image is the same size as the original, but is reflected across the x-axis.

c. When $a = -0.5$ and $b = -0.5$, the image is half the size as the original, both vertically and horizontally. However, it is rotated by 180° about the origin. (This rotation is equivalent to two reflections, across each of the axes.)

Stretches and shrinks by negative values result in reflection about the corresponding axis.

This presentation lays the groundwork for using stretches and shrinks to transform functions. First remind students how dilation works in geometry. Explain that the transformations involved in this presentation are different, because dilation involves equal stretches or shrinks both vertically and horizontally, but here they will see stretches and shrinks that are different in the two directions.

STRETCH AND SHRINK

First you'll shrink or stretch to move a point toward or away from the *x*-axis.

1. Open **Stretch Shrink Coords Present.gsp.** The sketch contains point *A*.
2. Press the buttons to measure *A*'s coordinates and to multiply the *y*-value by 2.
3. Press *Construct Image of A* to plot a new point using the calculated *y*-value.

Q1 Drag point *A* to a variety of different locations, and ask students to describe the behavior they observe.

Now make the multiplier adjustable by using the *a* slider.

4. Press *Show Slider* to make the slider visible, and press *Use Slider as Multiplier* to use it in the calculation in place of the constant 2.

Q2 Drag point *A* to several different locations, and ask students to describe how the image behaves. Then change the value of the slider and drag *A* again.

5. On page 2 the pre-image point *A* has been attached to a polygon. Drag the point and ask students to observe the behavior of its image A'. Press *Show Polygon Image* to show the path of the image point.

Q3 Adjust the *a* slider. Ask students to describe the shape and behavior of the image for various values of *a*.

6. On page 3 are two sliders, allowing you to transform the *x*- and *y*-values separately. Press the first few buttons to show the various elements and to show the image of the polygon.

Q4 Ask students to predict the image for each combination of slider values listed in the sketch. Then adjust the sliders to determine how accurate the predictions were. (All the combinations on this page have $a = b$, so the image is a similar shape to the pre-image.)

Pages 4 and 5 contain similar lists of slider values. If students can predict most of these correctly, they have a good sense of how stretches and shrinks work.

Transforming Coordinates

There are several Sketchpad activities in which you transform a point geometrically and then predict and observe the effect on the point's algebraic coordinates. In this activity you'll do the reverse: You'll transform the coordinates and observe the geometric effect on the plotted images.

GETTING STARTED

The coordinate system appears automatically.

1. In a new sketch, construct a point and measure its coordinates using **Measure | Abscissa (x)** and **Measure | Ordinate (y).**

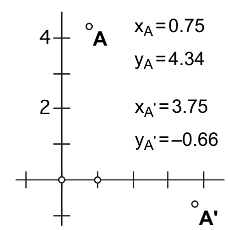
To put a measurement like x_A into a calculation, click the measurement in the sketch.

To change the label of a calculation, double-click it using the **Text** tool. Put the subscript in square brackets: x[A′] for $x_{A'}$.

2. To transform the coordinates algebraically, choose **Measure | Calculate,** and use the Calculator to define these two values:

$$x_{A'} = x_A + 3 \quad y_{A'} = y_A - 5$$

3. Plot the point determined by these transformed coordinates by selecting them in order and choosing **Graph | Plot As (x, y).** Label the new point A'.



Q1 The two formulas define the coordinates of image point A'. Drag A and describe how the transformation behaves. What type of transformation is this?

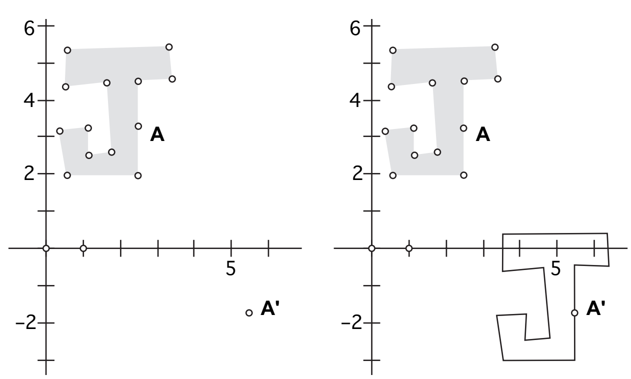
You can understand the transformation more easily by applying it to more than one point. Here you will apply it to a polygon.

Holding the Shift key while you click keeps all the points selected.

4. To construct the polygon, hold down the Shift key and use the **Point** tool to construct several points to be the vertices. While the points are still selected, choose **Construct | Polygon Interior.** Drag the vertices to give the polygon whatever shape you want.

5. To attach point A to the polygon boundary, select both A and the polygon, and choose **Edit | Merge Point To Polygon.**

6. Drag point A around the polygon and observe the path of A'. Then construct the locus of A' by selecting points A and A' and choosing **Construct | Locus.**



Now you can see the transformation more clearly. Drag the polygon vertices and watch how the image moves. Does this confirm your response to Q1?

Q2 You can change the definitions for the ordered pair $(x_{A'}, y_{A'})$. Double-click on a calculation in order to edit it. For each pair of definitions below, draw on your paper a prediction of what the polygon image will look like. Then change the coordinate definitions to check your prediction.

a. $(-x_A, y_A)$ b. $(-x_A, -y_A)$ c. $(-y_A, x_A)$

d. (y_A, x_A) e. $(2x_A, 2y_A)$ f. $(3x_A, y_A)$

Q3 Here are some transformations that were not included in the list above. State definitions for $(x_{A'}, y_{A'})$ that will create these transformations:

a. Reflect on the line $y = -x$.

b. Rotate 90° clockwise about the origin.

MORE DIFFICULT TRANSFORMATIONS

7. The coordinate definitions below produce a dilation by ratio 3 with respect to the point (2, 4). Edit the coordinates accordingly to confirm that this is correct.

$$x_{A'} = 2 + 3(x_A - 2) \quad y_{A'} = 4 + 3(y_A - 4)$$

Q4 Using a method similar to that in step 7, find the coordinate definitions for the following transformations:

a. Rotate 90° counter-clockwise about the point $(-5, 2)$.

b. Reflect across the line $y = 6$.

EXPLORE MORE

Q5 Construct a point in an arbitrary location, and use it as the center of dilation or rotation. The trick is to measure the coordinates of the center point, then use them in the calculations that define the coordinates of A'.

Q6 You can also use a slider or a parameter to define the ratio of dilation. Give that a try.

Q7 Each of the coordinate definitions described here contained only one variable (either x_A or y_A). Try a definition like $x_{A'} = x_A + 0.5y_A + 4$, $y_{A'} = y_A - 1$. Describe your results.

Transforming Coordinates

Objective: Students perform elementary transformations in the coordinate plane by applying mapping formulas to point coordinates.

Student Audience: Algebra 1/Algebra 2

Prerequisites: It would be best if students first have experience graphically rendering translations, reflections, right angle rotations, and dilations in the coordinate plane.

Sketchpad Level: Intermediate

Activity Time: 30–40 minutes

Setting: Paired/Individual Activity (no prepared sketch) or Whole-Class Presentation (use **Coordinate Transformation Present.gsp**)

The presentation document has constructions for all of the transformations in Q1 and Q2. If there is a need to save time, or if the students have not acquired the necessary Sketchpad skills, they can use this prepared document.

GETTING STARTED

2. If students are not already familiar with the Calculator, demonstrate the process of clicking on measurements in the sketch to insert them into a calculation.

Q1 This is a translation by vector $(3, -5)$.

Q2
a. Reflection across the y-axis
b. Rotation by 180° about the origin
c. Rotation by 90° counter-clockwise about the origin
d. Reflection across the line $y = x$
e. Dilation with respect to the origin by ratio 2
f. Horizontal stretch from the y-axis by ratio 3

Q3
a. $x_{A'} = -y_A, \quad y_{A'} = -x_A$
b. $x_{A'} = y_A, \quad y_{A'} = -x_A$

MORE DIFFICULT TRANSFORMATIONS

Q4
a. $x_{A'} = -5 - (y_A - 2), \quad y_{A'} = 2 + (x_A + 5)$
b. $x_{A'} = x_A, \quad y_{A'} = 12 - y_A$

EXPLORE MORE

Q5 Examples of dilation and rotation about arbitrary points are shown on pages B and C of the presentation file, **Coordinate Transformation Present.gsp.**

Q6 Pages B and C of the presentation file also contain sliders to define the ratio or angle.

Q7 The example given in the question describes a translation combined with a shear, and is illustrated on page D of the presentation file. (That page also contains a slider to allow you to adjust the shear.) Calculations that use linear combinations of x_A and y_A make possible the full set of affine transformations, including translation, rotation, reflection, dilation, stretching, and shear. An *affine* transformation is one that preserves collinearity and ratios of distances. In other words, any affine transformation of a segment with a midpoint results in another segment with a midpoint: The segment remains straight (collinearity) and the midpoint remains in the middle (ratio of distances).

Other activities are available to show the effects that geometric transformations have on coordinates. In this presentation you will turn it around by transforming the coordinates in order to transform the image.

1. Open **Coordinate Transformation Present.gsp.** Drag point *A* and ask students to observe its coordinates.

Q1 Point out the two calculations in blue. The first one adds 3 to x_A, and the second one subtracts 5 from y_A. These will be the coordinates of a new point. Where will it appear in relation to point *A*? (3 units to the right and 5 down)

2. Select the blue coordinates in order and choose **Graph | Plot As (x, y).** Label the new point A'. Drag point *A* again so students can see that A' moves with it.

Q2 Press the *Show Polygon* button. What would the result be if you added 3 to the *x*-coordinate and subtracted 5 from the *y*-coordinate of every point on the polygon? (The entire polygon would be translated.)

3. Select point *A* and the polygon interior. Choose **Edit | Merge Point To Polygon Interior.** Animate or drag point *A* so that it moves around the polygon.

Q3 As point *A* is moving, what is A' doing? (It is tracing a congruent polygon.)

4. Select points *A* and A' and choose **Construct | Locus.** The translated polygon image appears. Drag vertices of the pre-image polygon to help show the relationship.

The pages labeled (a)–(f) correspond to Q2 from the student activity. In each case point *A* is already attached to the polygon. The transformation has been constructed, but it is hidden. Stop at each page and challenge the class to describe the coordinate transformation. Use the *Show* buttons to show the results.

Q4 Move to the next page. Where will the polygon image be? Press the *Show Image* button to see the transformed polygon.

Q5 Ask for a precise geometric description of the transformation. Press the *Show Description* button to check the responses.

The pages labeled A, B, C, and D are considerably more complex. The first three involve transformations with respect to points other than the origin, and page D involves a shear. A slider controls the scalar *r*, and a dial controls angle *q*.

The final page, labeled "Custom," may be used to create your own transformations. Describe a transformation and ask students how to define it. Edit the blue coordinate calculations according to their directions.

Translating Functions

When you analyze a function, it helps to be able to picture its graph without necessarily drawing it. This is not easy if the function definition is complicated, but you may be able to recognize it as a translation of a simpler graph.

ADD TO THE FUNCTION

1. Open **Translating Functions.gsp.** You will see the function graph $y = f(x)$. The function itself is not important to this investigation. It's just a curve with plenty of ups and downs, giving it a distinctive shape. Point P is attached to the x-axis. The measurement x_p is the x-coordinate of P.

In this section you will transform the function by adding a constant to the value of the function itself.

Enter the function *f* by clicking the function definition on the screen.

2. Choose **Measure | Calculate** and calculate $f(x_p)$.
3. Select in order x_p and $f(x_p)$. Choose **Graph | Plot As (x, y).** A new point appears. Label it A.

Q1 Drag point P along the x-axis. What is the path of point A?

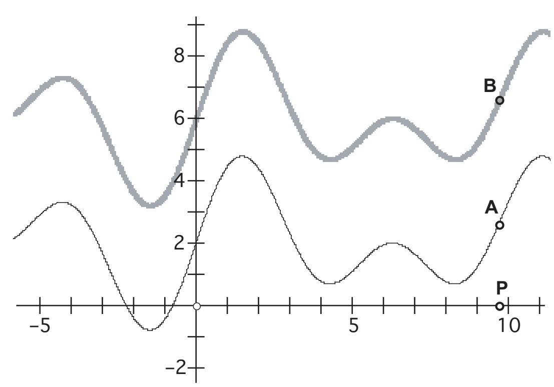
Now you will see what happens graphically when you add a constant to the function.

4. Using the Sketchpad Calculator again, compute the value $f(x_p) + 4$. Plot point B with coordinates $(x_p, f(x_p) + 4)$.

Q2 Where is point B in relation to point A? Is this relationship the same for all values of x? Test this by dragging point P.

Q3 Describe the path of B as you drag point P.

5. Select B and choose **Display | Trace Plotted Point.** Drag P again to trace the path of B.



Q4 The curve you have just traced is $y = f(x) + 4$. Describe it as a translation of $y = f(x)$.

To erase the old traces, press the *Erase Traces* button at the bottom of the screen.

6. Double-click the calculation $f(x_p) + 4$. Change it to $f(x_p) - 7$. Drag point P and observe the path of B again. Try some other constants. Double-click the original function to change its definition. Make up a function of your own.

Q5 From your observations, what general conclusion can you draw about the shape of a graph in the form $y = f(x) + k$?

SUBTRACT FROM THE ARGUMENT

You just saw what happens when you add a constant to the function. In this section you will subtract a constant from the argument (x).

7. Go to page 2. This should look familiar. It's the same as page 1.
8. Calculate $x_P - 2$ and $f(x_P - 2)$. Plot the point $(x_P - 2, f(x_P - 2))$. Label the point A.

Q6 Where does point A appear?

9. Plot the point $(x_P, f(x_P - 2))$. Label it B.

Q7 Where does point B appear with respect to point A? Explain why this is so. Drag point P to confirm that this relationship is always true.

Q8 As you drag point P, changing x, what is the path of point B? Trace point B to confirm this.

10. Experiment with subtracting other constants (positive and negative) from the argument. Try several different functions.

Q9 From your observations, what general conclusion can you make about the shape of a graph in the form $y = f(x - h)$?

SUMMARY

11. Open page 3. The two sliders control the parameters h and k.
12. Choose **Graph | Plot New Function.** Define the function $g(x) = f(x - h) + k$.
13. Drag each slider in turn and observe its effect.

Q10 Describe the graph of $y = f(x - h) + k$ as a transformation of $y = f(x)$.

Translating Functions

Objective: Students translate function graphs vertically and horizontally by adding constants to the function and the function argument.

Student Audience: Algebra 2

Prerequisites: Students must be familiar with function notation and graphing.

Sketchpad Level: Intermediate. Most of the work involves editing calculations and function definitions.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Translating Functions.gsp**) or Whole-Class Presentation (use **Translating Functions Present.gsp**)

ADD TO THE FUNCTION

Q1 The path of A is $y = f(x)$, the function plot.

Q2 Point B is four units above A. This relationship holds no matter where point P is.

Q3 Since B is always four units above A, and A is always on the function graph, the path of B has the same shape as the function graph and is four units higher.

Q4 The graph of $y = f(x) + 4$ is the graph of $y = f(x)$ translated four units upward.

Q5 For any function $f(x)$ and any constant k, the graph of $y = f(x) + k$ is the same as that of $y = f(x)$ translated k units upward. If $k < 0$, the translation is downward.

SUBTRACT FROM THE ARGUMENT

Q6 Point A falls on the function graph and is two units to the left of point P.

Q7 Points A and B are at the same height because their y-coordinates are the same. The x-coordinate of B is two units greater than that of A, so B is two units to the right of A.

Q8 Since B is always two units to the right of A, and A is always on the function graph, the path of B has the same shape as the function graph and is two units to the right of the function graph.

Q9 For any function $f(x)$ and any constant h, the graph of $y = f(x - h)$ is the same as the graph of $y = f(x)$ translated h units to the right. If $h < 0$, the translation is leftward.

SUMMARY

Q10 For any function $f(x)$ and constants h and k, the graph of $y = f(x - h) + k$ is the same as $y = f(x)$ translated h units to the right and k units upward.

In this presentation you will demonstrate the function translations of a graph in the form $y = f(x - h) + k$ by showing the vertical and horizontal translation components separately.

1. Open **Translating Functions Present.gsp.** Drag point P to show that it controls the measurement x_P. Drag the slider to show changes to parameter k.
2. Press *Show A*. Point A has coordinates $(x_P, f(x_P))$, so it must fall on the graph.

Q1 Drag the slider so that $k = 3$. For now, you are interested in seeing what happens when you add some constant to the function. If you plot the point $(x_P, f(x_P) + k)$, where will it fall? (It will be 3 units above point A.)

3. Press *Show B*. Drag P to show that B is always three units higher.

Q2 What is the path of point B as you drag P left and right? (Its path is a translation of the graph of $y = f(x)$, 3 units upward.)

4. Select point B and choose **Display | Trace Plotted Point.** Drag point P to trace the path.

Q3 What is the equation of this curve? ($y = f(x) + 3$. Show it with other values of k by dragging slider k to a new value and then dragging P again.)

5. Go to page 2 and show point A. This time it has coordinates $(x_P - h, f(x_P - h))$. This again places it on the graph.

Q4 Drag the slider so that $h = 4$, and plot $(x_P, f(x_P - h))$. Where will it fall? (It will be 4 units to the right of point A.)

6. Press *Show B* and drag point P left and right.

Q5 What is its path? (It is a translation of $y = f(x)$, 4 units to the right.)

7. Trace point B to demonstrate the shape of the path.

Q6 What is the equation of its path? ($y = f(x - h)$)

8. Page 3 has sliders for both h and k. The combined graph $y = f(x - h) + k$ is plotted. There are Show/Hide buttons for it. Try several different combinations for the two parameters, and challenge students to describe the translation before you reveal it in the sketch.
9. Double-click the definition for $f(x)$ to edit it. Try several different definitions. Students need to understand that the nature of the parent function has nothing to do with the translation principle.

Reflecting Functions

If you explored what happens when you reflect points across the *x*- and *y*-axes, you learned that the *x*-coordinates of a point and its reflection across the *y*-axis are the opposite of each other. Similarly, the *y*-coordinates of a point and its reflection across the *x*-axis are opposites. These rules allow you to reflect points across the axes. But what if you want to reflect an entire *function plot* across one of the axes?

COMPARE REFLECTIONS

In the New Function Calculator, use the **^** key to enter exponents.

To enter *f* into a new function equation, click its equation in the sketch.

To begin, plot a function and create two other functions related to the first.

1. In a new sketch, plot $f(x) = x^2 - 5x + 5$ using **Graph | Plot New Function.**
2. The **Graph | New Function** command allows you to enter a function without plotting it. Use this command to create the function $g(x) = -f(x)$. Then create the function $h(x) = f(-x)$.

The graph of one of these new functions will be the reflection of the original graph across the *y*-axis (labeled *A* in the figure below); the graph of the other will be the reflection across the *x*-axis (labeled *B* in the figure below).

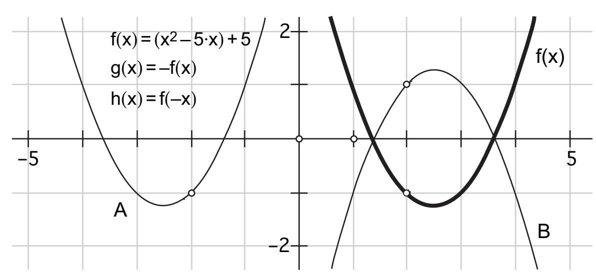
Q1 Do you think that $g(x)$ will match plot *A* or plot *B*? Which plot will $h(x)$ match? Explain why you paired them up the way you did.

To plot a function, select its equation and choose **Graph | Plot Function.**

Q2 Plot the two new functions (first one, then the other, so you know which is which). Was your pairing correct?

3. To make them easy to identify, choose different colors for $f(x)$, $g(x)$, and $h(x)$. (Use **Display | Color,** and color both the equation and the plot.) Also make the plot of $f(x)$ thick. (Use **Display | Line Weight | Thick.**)

Q3 If $x = 2$, what is $f(x)$? Write the result as an ordered pair.



To construct the point, choose the **Point** tool and click on the Function plot.

4. Construct a point on the function plot of *f* at the correct position for this ordered pair. Then construct a corresponding point on the function plot of *g*, and another on the function plot of *h*.

Q4 What are the coordinates of the corresponding point on $g(x)$? Explain why this makes sense algebraically, based on the definition that $g(x) = -f(x)$.

Q5 What are the coordinates of the corresponding point on $h(x)$? Explain why this makes sense algebraically, based on the definition that $h(x) = f(-x)$.

5. Double-click the equation for $f(x)$. In the Edit Function dialog box that appears, change $f(x)$ into some other function—perhaps a different quadratic, an absolute value function, or another polynomial like $f(x) = x^3 - x^2$.

Q6 How does the new $f(x)$ relate to the new graphs of $g(x)$ and $h(x)$?

Q7 Fill in the blanks:

a. The graphs of $y = f(x)$ and $y = -f(x)$ are reflections of each other across ____________.

b. The graphs of $y = f(x)$ and $y = f(-x)$ are reflections of each other across ____________.

EXPLORE MORE

Q8 From studying transformations, you've probably learned that reflecting a point over the x-axis, then reflecting that image point over the y-axis gives the same result as reflecting first over the y-axis, then over the x-axis (and that both are equivalent to a 180° rotation). Can you devise a function based on any $f(x)$ that, when plotted, transforms the plot of $f(x)$ twice, in this way?

Q9 An *even function* is one for which $f(-x) = f(x)$. In other words, the plots of $y = f(-x)$ and $y = f(x)$ are the same. An *odd function* is one for which $f(-x) = -f(x)$. In other words, the plots of $y = f(-x)$ and $y = -f(x)$ are the same. Edit $f(x)$ in your sketch to determine which of the following functions are even, which are odd, and which are neither:

a. $f(x) = x^2 + 3$

b. $f(x) = x^3$

c. $f(x) = x^3 + 3$

d. $f(x) = x^6 + 3x^4 - 2x^2$

e. $f(x) = x^5 + 3x^3 - 8x$

Q10 When you reflect a point across the line $y = x$, its x- and y-coordinates are switched. Here's a way of switching the x- and y-values for an entire plot. Choose **Graph | Plot New Function.** Choose **x = f(y)** from the Equation pop-up menu. Enter $f(y)$ (by clicking first on the equation of $f(x)$ in the sketch, then on y in the Calculator keypad), and click OK. Does the resulting graph appear to be a reflection of the original graph across $x = y$? Experiment with several different equations for $f(x)$ to see how a graph relates to its reflection across $y = x$.

Objective: Students reflect function plots across the x- and y-axes and explore connections between algebraic and geometric transformations.

Student Audience: Algebra 1/Algebra 2

Prerequisites: Students should be comfortable with function notation and function plotting (graphing parabolas, for example). They should also have a general understanding of geometric reflection and transformations.

Sketchpad Level: Intermediate

Activity Time: 15–30 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Reflecting Functions Present.gsp**)

Related Activities: Reflecting in Geometry and Algebra, Transforming Odd and Even Functions

A key goal of this activity is for students to understand the notation $y = f(x)$. Specifically, it should help them understand that the part inside the parentheses refers to the input (the x-value), and that using the opposite of x as the input alters its horizontal aspect, so the graph of $f(x)$ is reflected across the y-axis. By contrast, the entire expression $f(x)$ refers to the output (the y-value), and taking the opposite of the output affects the vertical aspect of the graph, so the graph of $f(x)$ is reflected across the x-axis. Students will find this easier to grasp if they start out with a good understanding of function notation and transformations.

COMPARE REFLECTIONS

Q1 Answers will vary. The important thing is that students make a guess and try to explain why.

Q2 Try asking a few students who got the correct pairing to explain their guesses, even if they were based only on a hunch. Also remember to congratulate those students who admit that they guessed wrong; emphasize that we learn best if we're willing to make mistakes.

Q3 If $x = 2$, $f(x) = -1$. The ordered pair is $(2, -1)$.

Q4 The corresponding point on g is $(2, 1)$.This makes sense because the value of $g(x)$ is the opposite of the value of $f(x)$, and 1 is the opposite of -1.

Q5 The corresponding point on h is $(-2, -1)$. By definition, $h(x) = f(-x)$. Therefore, $h(-2) = f(2)$.

Q6 Students should observe that, no matter how they change the function, $g(x)$ still reflects $f(x)$ across the x-axis and $h(x)$ across the y-axis.

Q7 a. The graphs of $y = f(x)$ and $y = -f(x)$ are reflections of each other across the x-axis.

b. The graphs of $y = f(x)$ and $y = f(-x)$ are reflections of each other across the y-axis.

EXPLORE MORE

Q8 The function $y = -f(-x)$ does the trick. The negative sign outside of $f(x)$ reflects f across the x-axis, and the one on the inside reflects it across the y-axis, as explained in the previous answer.

Q9 a. $f(x) = x^2 + 3$ is even.

b. $f(x) = x^3$ is odd.

c. $f(x) = x^3 + 3$ is neither.

d. $f(x) = x^6 + 3x^4 - 2x^2$ is even.

e. $f(x) = x^5 + 3x^3 - 8x$ is odd.

Q10 The plot of $x = f(y)$ is not necessarily equivalent to the plot of the inverse function $y = f^{-1}(x)$. To make an inverse function, you may need to restrict the domain of $f(x)$.

In this presentation students will discover how the graphs of $y = f(x)$ and $y = -f(x)$ are reflections of each other across the x-axis, and how the graphs of $y = f(x)$ and $y = f(-x)$ are reflections of each other across the y-axis. They will also learn how to identify functions as odd or even by observing their reflections.

1. Open **Reflecting Functions Present.gsp.** Press *Show f(x),* and ask students to identify the type of function and the shape it plots.

Q1 Press *Show g(x) and h(x)*. Tell students that g and h are both reflections of $f(x)$; then ask them to predict which of the two is a reflection across the x-axis and which is a reflection across the y-axis.

Q2 Ask students for an ordered pair that belongs to $f(x)$. Then ask them for the corresponding ordered pair for g, and the corresponding ordered pair for h. (For instance, if the ordered pair for f is $(2, -1)$, the corresponding pair for g must have the opposite y-value—$(2, 1)$—and the corresponding pair for h must have the opposite x-value—$(-2, -1)$.)

Q3 Ask, "How does the placement of the negative sign affect where the ordered pair will be graphed?" (If the ordered pair has the opposite x-value, it will be plotted on the opposite side of the y-axis. If it has the opposite y-value, it will be plotted on the opposite side of the x-axis.)

Q4 Press *Show Plots of g and h*, and ask students to use their predictions to match up the plots with their corresponding functions. (Function g matches plot *A,* and function h matches plot *B.*)

Q5 Press *Show Pairing* and ask students to evaluate their predictions.

2. Use the second set of functions to reinforce what students have just learned.

Algebraically, for an even function, $f(x) = f(-x)$. For an odd function, $-f(x) = f(-x)$.

3. Go to the "Odd or Even?" page. Explain that for an *even* function, the original function and its reflection across the y-axis are identical. For an *odd* function, the reflection across the x-axis is identical to the reflection across the y-axis.

Q6 For each of the five functions, first press the Show button for that function. Then ask students to decide whether the function is odd, even, or neither.

Q7 Press *Reflect Across x-axis,* and then *Reflect Across y-axis.* Ask students whether their hypotheses were correct. Ask, "How do these reflections help you tell?"

4. Repeat for the remaining functions. For each function, have students decide whether it's odd, even, or neither, both before and after showing the reflections.

Finish with a class discussion about how $-f(x)$ and $f(-x)$ are different from $f(x)$ and from each other. Encourage students to use both algebraic and geometric reasoning.

Stretching and Shrinking Functions

When you analyze a new function, it's easier to understand the function's behavior and graph if you can recognize it as a transformed version of a function you already know. In this activity you'll explore function transformations that involve stretching or shrinking a parent function.

MULTIPLY THE VALUE OF THE FUNCTION

1. Open **Stretching Functions.gsp.** You will see the function graph $y = f(x)$. The function itself is not important to this investigation. It's just a curve with a distinctive shape. Point P is attached to the x-axis. The measurement x_p is the x-coordinate of P.

In this section you will transform the function by multiplying the value of the function by a constant.

Enter function f by clicking the function definition on the screen and enter the value of x_p by clicking it on the screen.

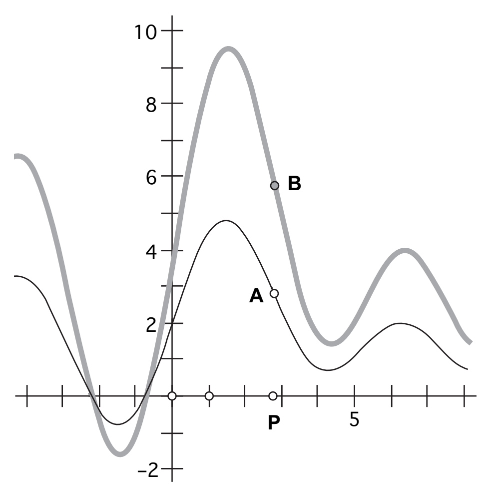
2. Choose **Measure | Calculate** and calculate $f(x_p)$.
3. Select in order x_p and $f(x_p)$. Choose **Graph | Plot As (x, y).** A new point appears. Label it A.

Q1 Drag point P along the x-axis. What is the path of point A?

Now you will see what happens graphically when you multiply the value of the function by a constant.

4. Using the Sketchpad Calculator again, compute the value $2 \cdot f(x_p)$. Then plot point B with coordinates $(x_p, 2 \cdot f(x_p))$.

Q2 Where is point B in relation to point A? Is this relationship the same for all values of x? Test this by dragging point P.



Q3 Describe the path of B as you drag point P.

5. Select B and choose **Display | Trace Plotted Point.** Drag P again to trace the path of B.

Q4 The curve you have just traced is $y = 2 \cdot f(x)$. Describe it as a transformation of $y = f(x)$.

To erase the old traces, press the *Erase Traces* button at the bottom of the screen.

6. Double-click the calculation $2 \cdot f(x_p)$. Change it to $0.5 \cdot f(x_p)$. Drag point P and observe the path of B again.

7. Show slider a. Change the calculation again, to make it $a \cdot f(x_p)$. Drag the slider and observe how point B behaves for different values of a.

To construct the locus, select both points and choose **Construct | Locus.**

8. To avoid the need to re-create traces every time you change the multiplier a, construct the locus of point B as P moves along the x-axis.

Q5 Use the slider to try various positive constants, some greater than 1 and some less than 1. Describe the transformation's effect on the graph when $a > 1$ and when $0 < a < 1$.

Q6 Try negative constants. Describe the transformation's effect on the graph when $a < -1$ and when $-1 < a < 0$. What happens when $a = 1$? When $a = 0$? When $a = -1$?

Q7 Double-click the original function to change its definition to a function of your own choice. Then try various values of a, and describe your results.

DIVIDE THE ARGUMENT OF THE FUNCTION

You just saw what happens when you multiply the function by a constant. In this section you will divide the argument (x) by a constant. You will take a shortcut by defining the transformed function directly, rather than transforming a point and constructing its locus.

9. Go to page 2. This page is similar to page 1, but the slider is labeled b. Set the value of b to 1.50.

To create the function, choose **Graph | Plot New Function.** To enter function f or value b into the new function, click the object in the sketch.

10. Create the transformed function in one step by graphing $g(x) = f(x/b)$.

Q8 Drag slider b to divide the argument by various constants. Try several different values, both positive and negative. Describe the effect on the transformed graph when $b > 1$, when $0 < b < 1$, when $b < -1$ and when $-1 < b < 0$.

Q9 Describe the shape of the transformed graph when $b = 1$, when $b = 0$, and when $b = -1$.

SUMMARY

11. Open page 3. The two sliders control the values a and b.

12. Define the function $g(x) = a \cdot f(x/b)$.

13. Drag each slider in turn and observe its effect.

Q10 Why do you think the directions asked you to *multiply* the value of the function by a, but to *divide* the argument by b?

Q11 Describe the graph of $g(x)$ as a transformation of $f(x)$.

Stretching and Shrinking Functions

Activity Notes

Objective: Students stretch and shrink function graphs vertically and horizontally by multiplying the function value and dividing the function argument by constants.

Student Audience: Algebra 2

Prerequisites: Students must be familiar with function notation and graphing.

Sketchpad Level: Intermediate. Most of the work involves editing calculations and function definitions.

Activity Time: 20–30 minutes

Setting: Paired/Individual Activity (use **Stretching Functions.gsp**) or Whole-Class Presentation (use **Stretching Functions Present.gsp**)

Related Activity: Function Transformation Game

MULTIPLY THE VALUE OF THE FUNCTION

Q1 The path of A is $y = f(x)$, the function plot.

Q2 Point B is twice as far from the x-axis as A. This relationship holds no matter where point P is.

Q3 Since B is always twice as far from the x-axis as A, and A is always on the function graph, the path of B is a vertically stretched version of the path of A.

Q4 The graph of $y = f(x) + 4$ is the graph of $y = f(x)$ stretched in such a way that the positive values are twice as high and the negative values are twice as low.

Q5 For any function $f(x)$ and any constant $a > 1$, the graph of $y = a \cdot f(x)$ is the same as that of $y = f(x)$ stretched vertically by a factor of a. If $0 < a < 1$, the transformed graph shrinks toward the x-axis.

Q6 Negative values of a flip the graph across the x-axis. When $a < -1$, the flipped graph also stretches vertically. When $-1 < a < 0$, the flipped graph shrinks vertically. When $a = 1$, the transformed graph is identical to the original. When $a = 0$, the transformed graph is flat, identical to the x-axis (a constant function with a value of zero). When $a = -1$, the transformed graph is the reflection of the original across the x-axis.

Q7 Answers will vary, because students will choose different functions. But no matter what functions they choose, they should see the same behavior that's described for the previous question.

DIVIDE THE ARGUMENT OF THE FUNCTION

Q8 When $b > 1$, the graph stretches horizontally, so that every point on it is farther from the y-axis. When $0 < b < 1$, the graph shrinks horizontally, so that every point on it is farther from the y-axis. When $b < -1$, the graph reflects across the y-axis and stretches horizontally. When $-1 < b < 0$, the graph reflects across the y-axis and shrinks horizontally.

Q9 When $b = 1$, the transformed graph is identical to the original. When $b = 0$, the transformed graph does not exist, because division by zero is undefined. When $b = -1$, the transformed graph is a reflection across the y-axis.

SUMMARY

Q10 When $a > 1$, multiplying the value of the function results in stretching vertically, because the value of the function becomes greater. When $b > 1$, dividing the argument of the function means that a greater value of the argument must be used for the same result, so this results in a horizontal stretch. (This phrasing assumes positive values, but the same principle applies if the value of the function or argument is negative.) By using multiplication in one case and division in the other, both operations result in similar behavior: a stretch when the parameter is greater than 1 and a shrink when it's between 0 and 1.

Q11 For any function $f(x)$ and constants a and b, the graph of $y = a \cdot f(x/b)$ is the same as the graph of $y = f(x)$ transformed vertically by a stretch or shrink by a factor of a, and transformed horizontally by a stretch or shrink by a factor of b. If either a or b is negative, the graph is also reflected across the x- or y-axis.

In this presentation you will demonstrate stretches and shrinks of a graph in the form $y = a \cdot f(x/b)$ by showing the vertical and horizontal stretches and shrinks separately on pages 1 and 2, and then combining them on page 3.

1. Open **Stretching Functions Present.gsp.** Drag point P to show that it controls the measurement x_P. Drag the slider to show how it changes parameter a.
2. Press *Show A*. This point has coordinates $(x_P, f(x_P))$, so it must fall on the graph. Drag the slider so that $a = 2$.

Q1 What will happen if you multiply the function's value by a? If you plot the point $(x_P, a \cdot f(x_P))$, where will it fall? (It will be twice as far above the x-axis as point A.)

3. Press *Show B*. Drag P to show that B is always twice as far from the axis as A.

Q2 What will be the path of point B as you drag P left and right? (Its path is a vertical stretch of the graph of $y = f(x)$ by a factor of 2.)

4. Select point B and choose **Display | Trace Plotted Point.** Drag point P to trace the path.

Q3 What is the equation of this curve? ($y = 2f(x)$. Show it with other values of a by dragging slider a to a new value and then dragging P again.)

5. Go to page 2 and drag the slider so that $b = 2$.

Q4 Now you want to plot $(x_P, f(x_P/b))$. Should you evaluate the function at point P, using the argument x_P? (No, you need to divide x_P by 2 before evaluating.)

6. Press *Show A* to show the argument to use when evaluating the function. Press *Show B* to evaluate the function using this argument.

Q5 Is this the point you want to plot for the new function? (No, this gives the correct value to use, but students need to plot this value at the position of P on the x-axis.)

7. Press *Show C* to plot the value of the function at the position of P. Drag P back and forth, and have students observe how the graph is stretched horizontally.
8. Page 3 has sliders for both a and b. The combined graph $y = a \cdot f(x/b)$ is plotted. Try several different combinations for the two parameters, and challenge students to describe the stretch or shrink before you reveal it in the sketch.
9. Double-click the definition of $f(x)$ to edit it. Try several different definitions. Students need to understand that the principle of stretching or shrinking a function is independent of the specific choice of the parent function.

Transforming Odd and Even Functions

An *odd function* is one in which $f(-x) = -f(x)$ for all x in its domain.

An *even function* is one in which $f(-x) = f(x)$ for all x in its domain.

In this activity you'll transform a point using the above definitions and use the transformed point to test a variety of functions to see if they are odd or even.

TRANSFORM A POINT

1. In a new sketch, construct point A and measure its x- and y-coordinates.

Next transform these coordinates according to the definition of an odd function.

2. The transformed point must have an x-value opposite that of the original point, so calculate $-x_A$.

Q1 From the definition of an odd function, what calculation can you use to find the y-value that corresponds to an x-value of $-x_A$? Calculate this result.

To plot the transformed point, select the calculated x- and y-values in order and choose **Graph | Plot As (x, y).**

3. Plot the transformed point and label the new point *Odd.*

Q2 From the definition of an even function, what calculations can you use to find the coordinates of a point that matches that rule?

4. Use the Calculator to compute these values. Plot the point and label it *Even.*

Q3 Describe the behavior of image points *Odd* and *Even* as you drag A.

TEST SOME FUNCTIONS

Select point A and the graph. Choose **Edit | Merge Point To Function Plot.**

5. Plot the function $f(x) = x^3 - 3x$. Merge point A to the new function plot.
6. Drag point A along the function plot and observe the transformed images.

Q4 How do the image points behave in relation to the function plot? Is the function even or odd? How can you tell from the image points?

Q5 Edit the function according to each problem below. Predict whether each will turn out to be even, odd, or neither. Then test by dragging A.

a. $f(x) = 5x$ b. $f(x) = x^3 + 2$ c. $f(x) = x^2 + 2$

d. $f(x) = x^4 - 3x^3$ e. $f(x) = \sin(x)$ f. $f(x) = \cos(x)$

Are there any functions that have odd symmetry about one point and even symmetry about a different point?

Q6 Some functions are neither odd nor even, but show even or odd symmetry about a point other than the origin. For instance, $f(x) = x^3 - 1$ shows odd symmetry about the point $(0, -1)$. Find two more such functions, one showing odd symmetry and one showing even. Identify the point or axis of symmetry, and edit the coordinate calculations to match the symmetry of the function.

Objective: Students transform the coordinates of a point according to the definitions of odd and even functions, and use the resulting images to explore the symmetry shown by the graphs of various specific functions.

Student Audience: Algebra 2

Prerequisites: None

Sketchpad Level: Challenging. Students should be familiar with the **Point** tool, with using the **Text** tool to label objects, with the Calculator, and with several commands from the Measure menu (**Abscissa** and **Ordinate**) and from the Graph menu (**Plot As (x, y)** and **Plot New Function**).

Activity Time: 25–35 minutes

Setting: Paired/Individual Activity (no sketch required) or Whole-Class Presentation (use **Transform Odd Even Present.gsp**)

TRANSFORM A POINT

1. Students must measure the x- and y-coordinates separately to use the results in calculations later.

Q1 In an odd function, the value of $f(-x)$ is the opposite of the value of $f(x)$, so the calculation should be $-y_A$.

3. The two calculations to plot are $(-x_A, -y_A)$.

Q2 In an even function, the value of $f(-x)$ is the same as the value of $f(x)$, so the values to plot are $(-x_A, y_A)$.

4. The two calculations to plot are $(-x_A, y_A)$.

Q3 Point *Even* is the reflection of A across the y-axis, and point *Odd* is its rotation by 180° about the origin. Students may also describe *Odd* as a double reflection across both x- and y-axes, as a dilation by -1 about the origin, or as a point reflection through the origin.

TEST SOME FUNCTIONS

Q4 As you drag A along the function plot, the *Odd* image traces out the opposite portion of the function plot. The *Even* image does not stay on the plot. This indicates that this particular function is odd, because the *Odd* image stays on it no matter where you drag A.

Q5 In each case students should test the function by dragging A along its plot. If the *Odd* image stays on the graph, the function is odd; if the *Even* image stays on the graph, the function is even; and if neither stays on it, the function is neither odd nor even. Here are the results:

a. $f(x) = 5x$ is odd.

b. $f(x) = x^3 + 2$ is neither.

c. $f(x) = x^2 + 2$ is even.

d. $f(x) = x^4 - 3x^3$ is neither.

e. $f(x) = \sin(x)$ is odd.

f. $f(x) = \cos(x)$ is even.

Q6 There are many possible answers.

Linear functions show point symmetry and are odd. (Any point on the graph can serve as the center.)

Second-degree (quadratic) polynomials are symmetric about a vertical axis and are even. (If the polynomial is expressed in the form $f(x) = a(x - h)^2 + k$, the axis of symmetry is the line $x = h$.)

Third-degree (cubic) polynomials are symmetric about a point and are odd. All these functions have a point of inflection, which is the point of symmetry.

Higher-degree polynomials generally exhibit neither kind of reflection symmetry. But some carefully chosen polynomials may show either kind of symmetry, depending on whether their highest-degree term has an even or odd exponent.

Absolute value functions have an axis of symmetry, and so exhibit even symmetry. Exponential and log functions exhibit neither odd nor even symmetry.

Sine and cosine graphs show both kinds of symmetry, requiring only a proper choice of point (for point symmetry) or vertical axis (for reflection symmetry). Because of their periodic property, these functions have infinitely many reflection points and reflection axes.

WHOLE-CLASS PRESENTATION

Use **Transform Odd Even Present.gsp** to present this activity to the class. Press the buttons and follow the directions on the screen to do the presentation.